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A SECOND ORDER IN TIME MODIFIED LAGRANGE–GALERKIN FINITE ELEMENT METHOD FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

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Dedicated to Professor J. Ildefonso Díaz on the occasion of his 60th birthday

Abstract. We introduce a second order in time modified Lagrange–Galerkin (MLG) method for the time dependent incompressible Navier–Stokes equations. The main ingredient of the new method is the scheme proposed to calculate in a more efficient manner the Galerkin projection of the functions transported along the characteristic curves of the transport operator. We present error estimates for velocity and pressure in the framework of mixed finite elements when either the mini-element or the $P2/P1$ Taylor–Hood element are used.

Key words. Lagrange–Galerkin, Navier–Stokes equations, mixed finite elements

AMS subject classifications. 65M12, 65M25, 65M60

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1. Introduction. The Lagrange–Galerkin (LG) method was introduced in the early 1980s by [6], [17], and [8] (see also [9]) to calculate a numerical solution of time dependent convection-diffusion problems, including the incompressible Navier–Stokes equations, represented by a differential equation of the form

$$\frac{Dv}{Dt} + Av = f,$$

where $\frac{D}{Dt} := \frac{\partial}{\partial t} + u \cdot \nabla$, u being a flow velocity, is the so-called transport operator, and A is a second order elliptic operator modeling the diffusion mechanism. The idea of this method is to combine an implicit backward in time discretization of the differential equation, along the characteristic curves of the transport operator, with a Galerkin projection in the framework of finite element methods (note that such an idea is also applicable in the context of spectral methods or hp-finite element methods; see, for instance, [19] and [10]), yielding in this way a marching in time procedure that may be very efficient for the following reasons: (i) the method partially circumvents the troubles caused by the convective terms because discretizing backward along the characteristic curves is a natural way of introducing upwinding in the space discretization of the differential equation; (ii) the resulting system of algebraic equations is symmetric and linear if the operator A is also, with a moderate condition number; (iii) the method is unconditionally stable if the Galerkin projection is performed exactly; this allows us to use a large time step Δt in the calculations.

Nevertheless, the LG method has several drawbacks: (i) the calculation of the feet of the characteristic curves at every time step; this requires solving, backward in

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time, many systems of ordinary differential equations (ODEs); and (ii) the calculation of some integrals, which come from the Galerkin projection, whose integrands are the product of functions defined in two different meshes. The first shortcoming is in some way related to the second because the integrals have to be computed exactly, but in general it cannot be done this way and they have to be numerically calculated with high accuracy to keep the method stable; see, in this respect, [2] where a study on the behavior of the method with different quadrature rules is performed. The use of high order quadrature rules means that many quadrature points per element should be employed to evaluate the integrals, and, therefore, since each quadrature point has an associated departure point, many systems of ODEs have to be solved numerically at every time step; hence, the whole procedure may become less efficient than it looks at first, in particular when working in unstructured meshes, because the numerical calculation of the feet of the characteristic curves requires locating and identifying of the elements containing such points, and this task is not easy to do in such meshes.

In [3] we introduced modified Lagrange–Galerkin (MLG) methods to partly overcome drawback (ii) of the conventional LG method while maintaining its rate of convergence when linear or quadratic finite elements are employed. The goal of this paper is to describe and analyze the convergence of an MLG method when it is applied to integrate the time dependent incompressible Navier–Stokes equations; in particular, we shall study the MLG method combined with the backward differentiation formula of order 2 (BDF2) as a time stepping scheme. The LG method combined with the BDF2 in a finite element context was presented for the first time in [7] to integrate convection diffusion problems; later on, [5] applied this method to integrate the incompressible Navier–Stokes equations.

We introduce some notation about the functional spaces we use in the paper. For $s \geq 0$ real and real $1 \leq p \leq \infty$, $W^{s,p}(D)$ denotes the real Sobolev spaces defined on D for scalar real-valued functions. $\|\cdot\|_{W^{s,p}(D)}$ and $|\cdot|_{W^{s,p}(D)}$ denote the norm and seminorm, respectively, of $W^{s,p}(D)$. When $s = 0$, $W^{0,p}(D) := L^p(D)$. For $p = 2$, the spaces $W^{s,2}(D)$ are denoted by $H^s(D)$, which are real Hilbert spaces with inner product $(\cdot,\cdot)_s$. For $s = 0$, $H^0(D) := L^2(D)$, the inner product in $L^2(D)$ is denoted by $(\cdot,\cdot)$. $H_0^1(D)$ is the space of functions of $H^1(D)$ which vanish on the boundary ∂D in the sense of trace. H^{-1} denotes the dual of $H_0^1(D)$. The corresponding spaces of real vector-valued functions, $v : D \to \mathbb{R}^d$, $d > 1$ integer, are denoted by boldface letters, for instance, $\mathbf{W}^{s,p}(D) := (W^{s,p}(D))^d := \{v : D \to \mathbb{R}^d : \ v_i \in W^{s,p}(D), \ 1 \leq i \leq d\}$. Let X be a real Banach space $(X, \|\cdot\|_X)$. If $v : (0,T) \to X$ is a strongly measurable function with values in X, we set $\|v\|_{L^p(0,t;X)} = (\int_0^t \|v(\tau)\|_X^p \, d\tau)^{1/p}$ for $1 \leq p < \infty$, and $\|v\|_{L^\infty(0,t;X)} = ess\sup_{0<\tau\leq t} \|v(\tau)\|_X$; when $t = T$, we shall write, unless otherwise stated, $\|v\|_{L^p(X)}$. We shall also use the following discrete norms:

$$\|v\|_{l^p(X)} = \left(\Delta t \sum_{i=1}^{N} \|v(\tau_i)\|_X^p\right)^{1/p}, \quad \|v\|_{l^\infty(X)} = \max_{1\leq i\leq N} \|v(\tau_i)\|_X .$$

Finally, we shall also make use of the space of continuous and bounded functions in time with values in X denoted by $C([0,T];X)$, and the space $C^{r,1}(\overline{D})$, $r \geq 0$, of functions defined in the closure of D, r-times differentiable and with the rth derivative being Lipschitz continuous.

The layout of the paper is as follows. In section 2 we introduce the continuous problem and its functional framework. In section 3 we describe the application of the MLG–BDF2 method to resolve the Navier–Stokes equations, using either the so-called mini-element or P_2/P_1 Taylor–Hood element. Section 4 is devoted to the numerical analysis of the method.

2. The continuous problem. Let $D \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded domain with smooth boundary ∂D, and let $[0, T]$ denote a time interval. For further information on the regularity hypotheses and the existence and uniqueness of the solutions of the Navier–Stokes equations, see [13] and [20]. In $Q_T := D \times (0, T)$ we consider the Navier–Stokes equations for a fluid of constant density ρ (for simplicity we take $\rho = 1$) under the action of an external force field $f(x, t)$ and with the known initial velocity, $u(x, 0) = u^0(x)$,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u + \nabla p - \nu \Delta u = f, \quad \text{div } u = 0, \quad u\,|_{\partial D} = 0, \tag{2.1}$$

where $u : \overline{D} \times [0, T) \to \mathbb{R}^d$ is the flow velocity, $p : D \times (0, T) \to \mathbb{R}$ is the pressure, and $f : D \times (0, T) \to \mathbb{R}^d$ denotes the density of the body forces per unit of mass. $\nu > 0$ is the kinematic viscosity coefficient, which is assumed to be constant. For the mathematical and numerical analysis of the solutions of (2.1) the following functional spaces are needed: $\mathbf{V} := \{u \in \mathbf{H}_0^1(D) :\ \text{div } u = 0 \text{ in } D\}$, $\mathbf{H} := \{u \in \mathbf{L}^2(D) :\ \text{div } u = 0 \text{ in } D \text{ and } n \cdot u = 0 \text{ on } \partial D\}$, where n is the unit outward normal to ∂D, $L_0^2(D) := \{q \in L^2(D) :\ \int_D q = 0.\}$. To calculate a numerical solution to (2.1) we use the following weak formulation [20].

Given $f \in L^2(\mathbf{H}^{-1})$ and $u^0 \in \mathbf{H}$, find $u \in L^2(\mathbf{H}_0^1(D)) \cap L^\infty(\mathbf{L}^2(D))$ and $p \in L^2(L_0^2(D))$ such that for all $v \in \mathbf{H}_0^1(D)$ and $q \in L_0^2(D)$,

$$\begin{cases} \dfrac{d}{dt}(u, v) + ((u \cdot \nabla)\, u, v) + \nu(\nabla u, \nabla v) - (p, \text{div } v) = (f, v), \\ \\ (\text{div } u, q) = 0. \end{cases} \tag{2.2}$$

2.1. Semidiscrete BDF2 Lagrangian formulation of the Navier–Stokes equations. To motivate the introduction of the MLG–BDF2 method, we present the BDF2 discretization of the Navier–Stokes equations, backward in time, along the characteristics of the operator $\frac{D}{Dt}$ in an interval $I_\tau := [\tau, s] \subset [0, T]$, $s > \tau$. To this end, we consider the mapping $x \in D \to X(x, s; t) \in D$, $t \in I_\tau$, defined by the initial value problem

$$\begin{cases} \dfrac{dX}{dt} = u(X(x, s; t), t), \\ \\ X(x, s; s) = x. \end{cases} \tag{2.3a}$$

If $u \in L^1(\tau, s; \mathbf{W}^{1,\infty}(D))$, this problem has a unique solution of the form

$$X(x, s; t) = x + \int_s^t u(X(x, s; \tau), \tau) d\tau. \tag{2.3b}$$

$t \to X(x, s; t)$ is a characteristic curve that represents the trajectory of a fluid particle that at time s will be at x. It is worth remarking that the mapping $X(x, s; t)$ has the group property; i.e., let t_1 and $t_2 \in I_\tau$, $t_1 < t_2$; then $X(x, s; t_1) = X(\cdot, t_2; t_1) \circ X(x, s; t_2)$. Hereafter, to simplify the writing of the formulas, unless otherwise stated, we adopt the notation $X^{k,l}(x) := X(x, t_l; t_k)$, k and l being positive integers. The following results, which are needed below, are well known.

LEMMA 2.1. *Assume that $u \in L^\infty(\mathbf{W}^{1,\infty}(D))$ and $s-\tau$ is sufficiently small; then $x \in D \to X(x,s;t)$ is a quasi-isometric homeomorphism of D onto D and its Jacobian determinant $J=1$ almost everywhere (a.e.) in D. Moreover,*

$$K_u^{-1}|x-z| \leq |X(x,s;\tau)-X(z,s;\tau)| \leq K_u|x-z|,$$

where $K_u = \exp((s-\tau)\|\nabla u\|_{L^\infty(0,T;\mathbf{L}^\infty(D))})$ and $|a-b|$ denotes the Euclidean distance between the points a and $b \in \mathbb{R}^d$.

For a proof of this lemma see [18]. In the following lemma we put together some facts concerning the solution of (2.3a) which are standard in the theory of ODE systems.

LEMMA 2.2. *Assume that $u \in L^\infty(\mathbf{W}^{k,\infty}(D))$, $k \geq 1$. Then for any integer n, $0 \leq n \leq N-1$, the unique solution $t \to X(x,t_{n+1};t)$ ($t \in [t_n,t_{n+1}] \subset [0,T]$) of (2.3a) is such that $X(x,t_{n+1};t) \in W^{1,\infty}(\mathbf{W}^{k,\infty}(D))$. Furthermore, let the multi-index $\alpha \in \mathbb{N}^d$; then for all α such that $1 \leq |\alpha| \leq k$, $\partial_{x_j}^{|\alpha|}X_i(x,t_{n+1};t) \in C([0,T];\mathbf{L}^\infty(D\times[0,T]))$, $1 \leq i,j \leq d$.*

Let $0=t_1<t_2<\cdots<t_N=T$ be a uniform partition of step Δt of the interval $[0,T]$. For x fixed let us consider the differential equation

$$\frac{dy(x,t)}{dt} = f(y,t), \quad t \in (0,T];$$

the BDF2 discretization of this equation at time t_{n+1} is of the form

$$d_t y(x,t_{n+1}) = f(y(x,t_{n+1}),t_{n+1}),$$

where

$$d_t y(x,t_{n+1}) := \frac{3y(x,t_{n+1})-4y(x,t_n)+y(x,t_{n-1})}{2\Delta t};$$

then, noting that $X^{n+1,n+1}(x)=x$, the BDF2 discretization of (2.1) along the characteristics curves in the interval $I_n=[t_n,t_{n+1}]$ is [5]

$$\text{(2.4)} \qquad \begin{cases} d_t u(X^{n+1,n+1}(x),t_{n+1}) + \nabla p(x,t_{n+1}) = \nu\Delta u(x,t_{n+1}) + f(x,t_{n+1}), \\ \text{div } u(x,t_{n+1}) = 0, \\ u(x,t_{n+1}) \mid_{\partial D} = 0, \end{cases}$$

where

$$d_t u(X^{n+1,n+1}(x),t_{n+1}) := \frac{3u(x,t_{n+1})-4u(X^{n,n+1}(x),t_n)+u(X^{n-1,n+1}(x),t_{n-1})}{2\Delta t}$$

is the BDF2 discretization of the total derivative $\frac{Du}{Dt}$.

3. The MLG–BDF2 method. In this section we describe the MLG–BDF2 method in a finite element framework. To do so, we introduce the finite element spaces, where the numerical solution is sought, and some of their properties needed for the analysis of the method.

3.1. Finite element spaces. We consider a family of regular quasi-uniform partitions D_h of the region $\overline{D}$ which are formed by simplices. However, if ∂D was not a polyhedral surface (polygonal line), it would be possible to use curved elements near the boundary: the element touching the boundary would have at least a curved face (side). (See [4] for the theory on curved elements.) As is usual in the finite element technique, we consider the reference element, $\widehat{T} := \{\widehat{x} \in \mathbb{R}^d : \ 0 \leq \widehat{x}_i \leq 1, \ 1 - \sum_{i=1}^d \widehat{x}_i \geq 0\}$, such that for each T_j there exists an invertible affine mapping $F_j : \widehat{T} \to T_j$ of the form

$$F_j(\widehat{x}) = \mathbf{B}_j \widehat{x} + \mathbf{b}_j, \quad \mathbf{B}_j \in \mathcal{L}(\mathbb{R}^d), \quad \text{and} \quad \mathbf{b}_j \in \mathbb{R}^d. \tag{3.1}$$

We associate with D_h the H^1-conforming finite element spaces $\mathbf{W}_h$ and $M_h \subset L_0^2(D)$. We shall approximate the velocity in $\mathbf{X}_h := \mathbf{W}_h \cap \mathbf{H}_0^1(D)$ and the pressure in M_h. Moreover, we assume that the spaces $\mathbf{X}_h$ and M_h have the following properties:

(P1) (Ladyzhenskaia–Babuška–Brezzi condition). There exists a positive constant β independent of the discretization parameter h such that

$$\inf_{q_h \in M_h} \sup_{v_h \in \mathbf{X}_h} \frac{(\text{div } v_h, q_h)}{\|v_h\|_{\mathbf{H}^1(D)} \|q_h\|_{L^2(D)}} \geq \beta. \tag{3.2a}$$

(P2) The elements of the spaces $\mathbf{X}_h$ and M_h are piecewise polynomials of degrees m and l, respectively; then assuming that $v \in \mathbf{H}^{s+1}(D) \cap \mathbf{H}_0^1(D)$, there exists a constant C_1 independent of h such that

$$\inf_{v_h \in \mathbf{X}_h} \left(\|v - v_h\|_{\mathbf{L}^2(D)} + h \|v - v_h\|_{\mathbf{H}^1(D)} \right) \leq C_1 h^{s+1} \|v\|_{\mathbf{H}^{s+1}(D)}, \quad 0 \leq s \leq m. \tag{3.2b}$$

(P3) Assuming that $p \in H^{s_1+1}(D)$, $0 \leq s_1 \leq l$, there exists a constant C_2 independent of h such that

$$\inf_{q_h \in M_h} \left(\|p - q_h\|_{L^2(D)} + h \|p - q_h\|_{H^1(D)} \right) \leq C_2 h^{s_1+1} \|p\|_{H^{s_1+1}(D)}. \tag{3.2c}$$

(P4) (inverse property). There exist positive constants C_3 and C independent of h such that for $v_h \in \mathbf{X}_h$,

$$\|v_h\|_{\mathbf{W}^{m,q}(D)} \leq C_3 h^{d/q - d/p + k - m} \|v_h\|_{\mathbf{W}^{k,p}(D)}, \quad 0 \leq k \leq m \leq 1, \quad 0 \leq p \leq q \leq \infty, \tag{3.2d}$$

and

$$\|v_h\|_{\mathbf{L}^\infty(D)} \leq D(h) \|v_h\|_{\mathbf{H}^1(D)}; \quad D(h) := \begin{cases} C(1 + |\log h|^{1/2}) & \text{if } d = 2, \\ C h^{-1/2} & \text{if } d = 3. \end{cases} \tag{3.2e}$$

Specifically, we shall consider the P_2/P_1 Taylor–Hood finite element and the so-called mini-element as examples of the spaces $(\mathbf{X}_h, M_h)$.

3.2. The formulation of the MLG–BDF2 method. We are ready to formulate the MLG–BDF2 method to approximate the weak solution of (2.4) in the finite element spaces $(\mathbf{X}_h, M_h)$. The statement of the method is as follows.

For $n = 1, 2, \ldots, N-1$, find $(u_h^{n+1}, p_h^{n+1}) \in \mathbf{X}_h \times M_h$ such that for any $v_h \in \mathbf{X}_h$ and $q_h \in M_h$ they are the unique solution to

$$\left\{\begin{array}{l} \left(\dfrac{3u_h^{n+1} - 4u_h^n(\widetilde{X}_h^{n,n+1}(x)) + u_h^{n-1}(\widetilde{X}_h^{n-1,n+1}(x))}{2\Delta t}, v_h\right) \\ \\ + \nu\left(\nabla u_h^{n+1}, \nabla v_h\right) - (p_h^{n+1}, \mathrm{div}\ v_h) = (f^{n+1}, v_h), \\ \\ \left(\mathrm{div}\ u_h^{n+1}, q_h\right) = 0. \end{array}\right. \tag{3.3}$$

Since $(u_h^{n-l}(\widetilde{X}_h^{n-l,n+1}(x)), v_h)=\sum_j \int_{T_j} u_h^{n-l}(\widetilde{X}_h^{n-l,n+1}(x)) \cdot v_h(x)dx$, $l = 0, 1$, the calculations of $\int_{T_j} u_h^{n-l}(\widetilde{X}_h^{n-l,n+1}(x)) \cdot v_h(x)dx$ and $\widetilde{X}_h^{n-l,n+1}(x)$ are key issues.

Remark 3.1. To initialize the calculations in (3.3) we need to know u_h^0 and (u_h^1, p_h^1). We assume that u_h^1 and p_h^1 are calculated by a second order in time single step scheme, and (u_h^0, p_h^0) by the scheme proposed in [1].

3.2.1. The points $\widetilde{X}_h^{n-l,n+1}(x)$, $l = 0, 1$. For $x \in D$, the points $\widetilde{X}_h^{n-l,n+1}(x)$ are approximations to the points $X_h^{n-l,n+1}(x)$ which are numerical solutions at time instants t_{n-l} of the initial value problem

$$\left\{\begin{array}{l} \dfrac{dX_h(x, t_{n+1}; t)}{dt} = u_h(X_h(x, t_{n+1}; t), t), \quad t_{n-l} \le t < t_{n+1}, \\ \\ X_h(x, t_{n+1}; t_{n+1}) = x, \end{array}\right. \tag{3.4}$$

where $u_h(x, t)$ is usually calculated by some extrapolation/interpolation formula of the values u_h^n and u_h^{n-1}. Note that $u_h(x, t)$ is in $C([t_{n-1}, t_{n+1}]; \mathbf{W}^{1,\infty}(D))$ because u_h^n, u_h^{n-1} are in $\mathbf{W}^{1,\infty}(D)$. Given the element T_j of D_h, we define the elements $T_{hj}^{n-l,n+1}$ (resp., $T_j^{n-l,n+1}$) by $T_{hj}^{n-l,n+1} := X_h^{n-l,n+1}(T_j)$ (resp., $T_j^{n-l,n+1} := X^{n-l,n+1}(T_j)$), and under the assumption of Lemma 2.2 we can define the quasi-isometric mappings $F_{hj}^{n-l,n+1} : \widehat{T} \to T_{hj}^{n-l,n+1}$ of class $\mathbf{C}^{0,1}$ and $F_j^{n-l,n+1} : \widehat{T} \to T_j^{n-l,n+1}$ of class $\mathbf{C}^{k-1,1}$, $k \ge 1$ integer, such that for all $\widehat{x} \in \widehat{T}$ and $x = F_j(\widehat{x}) \in T_j$

$$\left\{\begin{array}{l} F_{hj}^{n-l,n+1}(\widehat{x}) = X_h^{n-l,n+1} \circ F_j(\widehat{x}) = X_h^{n-l,n+1}(x), \\ F_j^{n-l,n+1}(\widehat{x}) = X^{n-l,n+1} \circ F_j(\widehat{x}) = X^{n-l,n+1}(x). \end{array}\right. \tag{3.5a}$$

In relation to the simplices $T_{hj}^{n-l,n+1}$, we also consider the simplices $\widetilde{T}_{hj}^{n-l,n+1}$ of vertices $\{X_h^{n-l,n+1}(a_1^{(j)}), \ldots, X_h^{n-l,n+1}(a_{d+1}^{(j)})\}$, with $\{a_i^{(j)}\}_{1 \le i \le d+1}$ being the vertices of the element T_j, and define the invertible affine mappings $\widetilde{F}_{hj}^{n-l,n+1} : \widehat{T} \to \widetilde{T}_{hj}^{n-l,n+1}$ such that

$$\widetilde{F}_{hj}^{n-l,n+1}(\widehat{x}) = \widetilde{\mathbf{B}}_{hj}^{n-l,n+1}\widehat{x} + \widetilde{\mathbf{b}}_{hj}^{n-l,n+1}, \tag{3.5b}$$

where $\widetilde{\mathbf{B}}_{hj}^{n-l,n+1} \in \mathcal{L}(\mathbb{R}^d)$ and $\widetilde{\mathbf{b}}_{hj}^{n-l,n+1} \in \mathbb{R}^d$. Similarly, for $T_j^{n-l,n+1}$ and $F_j^{n-l,n+1}$, we define the simplices $\widetilde{T}_j^{n-l,n+1}$, with vertices $\{X^{n-l,n+1}(a_1^{(j)}), \ldots, X^{n-l,n+1}(a_{d+1}^{(j)})\}$, and the invertible affine mappings $\widetilde{F}_j^{n-l,n+1} : \widehat{T} \to \widetilde{T}_j^{n-l,n+1}$ by

$$\widetilde{F}_j^{n-l,n+1}(\widehat{x}) = \widetilde{\mathbf{B}}_j^{n-l,n+1}\widehat{x} + \widetilde{\mathbf{b}}_j^{n-l,n+1}. \tag{3.5c}$$

Note that $\widetilde{T}_{hj}^{n-l,n+1}$ and $\widetilde{T}_{j}^{n-l,n+1}$ are linear approximations to $T_{hj}^{n-l,n+1}$ and $T_{j}^{n-l,n+1}$, respectively, and hence

$$\widetilde{F}_{hj}^{n-l,n+1} = \widehat{I}F_{hj}^{n-l,n+1} \quad \text{and} \quad \widetilde{F}_{j}^{n-l,n+1} = \widehat{I}F_{j}^{n-l,n+1}, \tag{3.5d}$$

where $\widehat{I}$ denotes the linear interpolant in $\widehat{T}$. Next, we define the mappings $\widetilde{X}_{j}^{n-l,n+1} : T_j \to \widetilde{T}_{j}^{n-l,n+1}$ and $\widetilde{X}_{hj}^{n-l,n+1} : T_j \to \widetilde{T}_{hj}^{n-l,n+1}$ by

$$\begin{aligned} \widetilde{X}_{j}^{n-l,n+1}(x) &= \widetilde{F}_{j}^{n-l,n+1}(\widehat{x}) = \widetilde{F}_{j}^{n-l,n+1} \circ F_j^{-1}(x), \\ \widetilde{X}_{hj}^{n-l,n+1}(x) &= \widetilde{F}_{hj}^{n-l,n+1}(\widehat{x}) = \widetilde{F}_{hj}^{n-l,n+1} \circ F_j^{-1}(x) \end{aligned} \tag{3.5e}$$

and construct the mappings $\widetilde{X}^{n-l,n+1} : \overline{D} \to \bigcup \widetilde{T}_{j}^{n-l,n+1}$ and $\widetilde{X}_{h}^{n-l,n+1} : \overline{D} \to \bigcup \widetilde{T}_{j}^{n-l,n+1}$ such that

$$\widetilde{X}^{n-l,n+1}(x) = \widetilde{X}_{j}^{n-l,n+1}(x), \; \widetilde{X}_{h}^{n-l,n+1}(x) = \widetilde{X}_{hj}^{n-l,n+1}(x) \quad \text{when } x \in T_j. \tag{3.5f}$$

We graphically show the construction of $\widetilde{X}_{j}^{n,n+1}(x)$ and $X_{j}^{n,n+1}(x)$ in Figure 3.1.

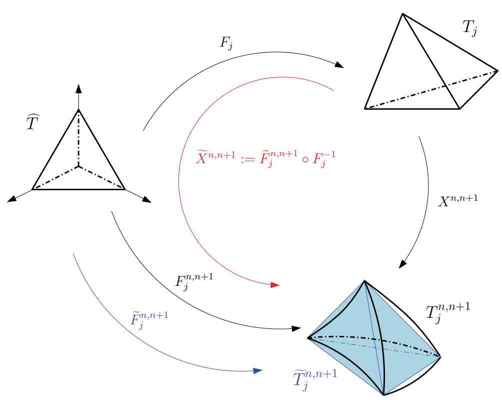

FIG. 3.1. *The points $\widetilde{X}_{j}^{n,n+1}(x)$ and $X_{j}^{n,n+1}(x)$ and the mappings for the formulation of the MLG method. $\widetilde{X}_{j}^{n,n+1}(x)$ is in the plane face tetrahedron and $X_{j}^{n,n+1}(x)$ is in the curved face tetrahedron.*

Remark 3.2. We note that if $\overline{D}$ is either a polygon or a polyhedron and T_b is a boundary element, i.e., $T_b \cap \partial D = \Gamma_b \subset \partial D$, then $T_b^{n-l,n+1} = X^{n-l,n+1}(T_b)$ and $\widetilde{T}_b^{n-l,n+1} = \widetilde{X}^{n-l,n+1}(T_b)$ are also boundary elements satisfying $T_b^{n-l,n+1} \cap \partial D = \widetilde{T}_b^{n-l,n+1} \cap \partial D = \Gamma_b$ because $u = 0$ on the boundary; for the same reasons, the intersection of $\widetilde{T}_{hb}^{n-l,n+1}$ with ∂D is also Γ_b; consequently, $\overline{D} = \cup \widetilde{T}_{j}^{n-l,n+1} = \cup \widetilde{T}_{hj}^{n-l,n+1}$. However, if ∂D were a curved boundary, then by construction, $T_b^{n-l,n+1} \cap \partial D = \Gamma_b$ would be a curved face which, in general, is different from the boundary face $\widetilde{\Gamma}_b^{n-l,n+1}$ of $\widetilde{T}_{hb}^{n-l,n+1}$ because the latter is a straight $d-1$ simplex. Nevertheless, $\widetilde{\Gamma}_b^{n-l,n+1}$ intersects Γ_b at its vertices. All the developments that follow are still valid for the case with curved boundary if one makes additional assumptions and uses the techniques of [4].

3.2.2. Calculation of $\int_{T_j} u_h^{n-l}(\widetilde{X}_h^{n-l,n+1}(x)) \cdot v_h(x)dx,\ l = 0,\ 1.$ The evaluation of the element integrals is usually done numerically by applying a quadrature rule of high order, so as to maintain both the stability and the accuracy that the method would possess if the integrals were calculated exactly. Since $u_h^{n-l} \in \mathbf{X}_h$,

$$u_h^{n-l}(x) = \sum_{i=1}^{M} U_i^{n-l}\phi_i(x),$$

where $\{\phi_i\}_{i=1}^M$ is the set of global basis functions of $\mathbf{X}_h$ and $U_i^{n-l} = u_h^{n-l}(x_i)$, with x_i being the ith mesh node. The restriction of u_h^{n-l} on the element T_j is written as

$$u_h^{n-l}(x)\mid_{T_j} = \sum_{k=1}^{ne} U_{k(j)}^{n-l}\varphi_k^{(j)}(x),$$

where ne is the number of velocity nodes in T_j, $k(j)$ denotes the global number of the node of the mesh D_h that is the kth node of T_j, and $\{\varphi_k^{(j)}\}_{k=1}^{ne}$ is the set of local basis functions for the element T_j. As is customary in the finite element technique, we employ the element of reference $\widehat{T}$ to calculate the integral over the element T_j. Thus, assuming that for $x \in T_j$, $\widetilde{X}_h^{n-l,n+1}(x)$ is in some simplex T_i of the fixed partition D_h, in general $T_i \neq T_j$, then $u_h^{n-l}(\widetilde{X}_h^{n-l,n+1}(x)) = \sum_{k=1}^{ne} U_{k(i)}^{n-l}\varphi_k^{(i)}(\widetilde{X}_h^{n-l,n+1}(x))$, and taking $v_h(x) = \varphi_p^{(j)}(x),\ 1 \leq p \leq ne$, we have

$$\int_{T_j} u_h^{n-l}(\widetilde{X}_h^{n-l,n+1}(x)) \cdot v_h(x)dx = \sum_{k=1}^{ne} U_{k(i)}^{n-l}\int_{T_j} \varphi_k^{(i)}(\widetilde{X}_h^{n-l,n+1}(x))\varphi_p^{(j)}(x)dx.$$

By (3.5e) and the assumption $\widetilde{X}_h^{n-l,n+1}(x) \in T_i$,

$$\int_{T_j} \varphi_k^{(i)}(\widetilde{X}_h^{n-l,n+1}(x))\varphi_p^{(j)}(x)dx = \int_{\widehat{T}} \widehat{\varphi}_k(\widehat{\widetilde{z}})\widehat{\varphi}_p(\widehat{x}) \left|\frac{\partial F_j}{\partial \widehat{x}}\right| d\widehat{x},$$

where $\widehat{\widetilde{z}} := F_i^{-1} \circ \widetilde{F}_{hj}^{n-l,n+1}(\widehat{x})$. Finally, we approximate the integrals over $\widehat{T}$ by high order quadrature rules as

$$\int_{\widehat{T}} \widehat{\varphi}_k(\widehat{\widetilde{z}})\widehat{\varphi}_p(\widehat{x}) \left|\frac{\partial F_j}{\partial \widehat{x}}\right| d\widehat{x} \simeq meas(T_j)\sum_{g=1}^{nqp} \varpi_g \widehat{\varphi}_k(\widehat{\widetilde{z}}_g)\widehat{\varphi}_p(\widehat{x}_g),$$

where nqp denotes the number of weights, ϖ_g, and points, $\widehat{x}_g$, of the quadrature rule.

Remark 3.3. Note that in order to calculate the integrals it is necessary to define the triangle $\widetilde{T}_{hj}^{n-l,n+1}$; this is done by computing at time t_{n-l} the points $\widetilde{X}_h^{n-l,n+1}(a_i^{(j)})$ as solutions of (3.4); this means that the number of departure points to be calculated every time step is NV, the number of vertex nodes, whereas in the conventional LG method such a number is $NE \times nqp$, which is much larger than NV because nqp is quite large in high order quadrature rules, in particular in three-dimensional problems. Since the integration of (3.4) and the identification of the element that contains the solution point are costly parts of the method, the MLG methods are more efficient in terms of CPU time than the conventional LG methods.

4. Convergence of the MLG–BDF2 method. In this section we perform the error analysis of the method following a step by step approach. First, we recall auxiliary results concerning the convergence of the semidiscrete Stokes problem; second, we study the error of the approximation of the departure points and some related results, considering that the system (3.4) is integrated by a Runge–Kutta scheme of order $r \geq 2$; third, we end up establishing the convergence of the method in the $l^\infty(\mathbf{L}^2(D))$ and $l^\infty(\mathbf{H}^1(D))$ norms for the velocity and $l^2(\mathbf{L}^2(D))$ norm for the pressure. In the developments that follow we need the finite dimensional space $\mathbf{V}_h$ defined as

$$\mathbf{V}_h = \{v_h \in \mathbf{X}_h : (\text{div } v_h, q_h) = 0 \ \ \forall q_h \in M_h\}.$$

Let $(u(t), p(t))$ be the weak solution to (2.1). We define $w_h : [0,T] \to \mathbf{X}_h$ and $\mu_h : [0,T] \to M_h$ as the solution of the semidiscrete Stokes problem

$$\text{(4.1)} \qquad \begin{cases} \nu(\nabla(u(t) - w_h(t)), \nabla v_h) - (p(t) - \mu_h(t), \text{div } v_h) = 0 \ \ \forall v_h \in \mathbf{V}_h, \\ \\ (\text{div}(u(t) - w_h(t)), q_h) = 0 \ \ \forall q_h \in M_h. \end{cases}$$

We have the following results (see [11, Chap 2]).

LEMMA 4.1. *Let $u \in L^\gamma(\mathbf{V} \cap \mathbf{H}^{s+1}(D))$ and $p \in L^\gamma(L_0^2(D) \cap H^{s_1+1}(D))$, $\gamma \in [1,\infty]$. Then there exist positive bounded constants C_7 and C_8 independent of Δt and h such that*

$$\text{(4.2)} \qquad \begin{aligned} &\|u - w_h\|_{L^\gamma(\mathbf{L}^2(D))} + h\left[\|u - w_h\|_{L^\gamma(\mathbf{H}^1(D))} + \|p - \mu_h\|_{L^\gamma(L^2(D))}\right] \\ &\leq \ C_7 h^{s+1}\|u\|_{L^\gamma(\mathbf{H}^{s+1}(D))} + \frac{C_8}{\nu} h^{s_1+2}\|p\|_{L^\gamma(H^{s_1+1}(D))}. \end{aligned}$$

Besides this lemma we need the following one that can be found in [12].

LEMMA 4.2. *Let the domain D be such that for $b > d$ and g in $\mathbf{L}^b(D)$, the solution (u,p) of the Stokes problem*

$$-\nu\Delta u + \nabla p = g, \quad \text{div } u = 0 \quad \textit{in} \quad D, \ u\,|_{\partial D} = 0$$

is in $\mathbf{W}^{2,b}(D) \times W^{1,b}(D)$ with continuous dependence on g, and assume that D_h is a quasi-uniform regular partition of D. If $(u,p) \in L^\gamma(\mathbf{W}^{1,\infty}(D)) \times L^\gamma(L^\infty(D))$, there is a constant C independent of h, u, and p such that

$$\text{(4.3)} \qquad \begin{aligned} &\|\nabla(u - w_h)\|_{\mathbf{L}^\infty(D)} + \|p - \mu_h\|_{L^\infty(D)} \\ &\leq \ C \ \inf_{(v_h,q_h)\in \mathbf{X}_h \times M_h}\left(\|\nabla(u - v_h)\|_{\mathbf{L}^\infty(D)} + \|p - q_h\|_{L^\infty(D)}\right). \end{aligned}$$

We introduce the notations $u_t := \frac{\partial u}{\partial t}$ and $D_t^k u := \frac{D^k u}{Dt^k}$, $k \geq 1$. For the Taylor–Hood elements, $l = m - 1$; specifically, for the P_2/P_1 element, $m = 2$ and $l = 1$. As for the mini-element, $l = 1$, whereas the polynomials for the velocity in an element T belong to $\mathbf{P}_1(T) \bigoplus \text{span}\Pi_{i=1}^{d+1}\varphi_i$; however, in the velocity error estimates for this element, $m = 1$; see [11]. To proceed with the analysis we state the following regularity hypotheses:

(R1) $u^0 \in \mathbf{H}^{m+1}(D) \cap \mathbf{W}^{2,\infty}(D) \cap \mathbf{V}$.

(R2) $u \in L^\infty(\mathbf{V} \cap \mathbf{H}^{m+1}(D) \cap \mathbf{W}^{2,\infty}(D)) \cap C(\mathbf{C}^{0,1}(\overline{D}))$, $u_t \in L^2(\mathbf{V} \cap \mathbf{H}^{m+1}(D))$; $D_t^2 u$; and $D_t^3 u \in L^2(\mathbf{L}^2(D))$; $p \in L^\infty(H^m(D) \cap L_0^2(D) \cap L^\infty(D))$; and $p_t \in L^2(H^m(D))$.

We also state the initial hypothesis (see Remark 3.1): for $l = 0, 1$,

$$\|u^l - u_h^l\|_{\mathbf{L}^2(D)} + h\left[\|u^l - u_h^l\|_{\mathbf{H}^1(D)} + \|p^l - p_h^l\|_{L^2(D)}\right] = O(h^{m+1} + l\Delta t^3). \tag{4.4}$$

As in [18] and [1], we prove the convergence of the method by induction on n and impose the mesh restriction $\Delta t = o(h^{d/4})$ to obtain optimal error estimates. Next, we state the *induction hypotheses* (IH).

(IH) *Let* $\Delta t = o(h^{d/4})$ *and assume that* (R1), (R2), *and the initial hypothesis holds; then for all* n *such that* $0 \le n < N$, *there exist constants* $h_s < 1$, $\overline{\varepsilon} = O(T)$, *and* $C > 0$ *independent of* Δt, h, *and* n *such that for* $h \in (0, h_s)$,

$$\|u - u_h\|_{l^\infty(0,t_n;\mathbf{L}^2(D))} \le C\left(h^{\omega_1} + \Delta t^2 + \frac{\sqrt{t_n}}{\Delta t}\min\left(\frac{K_4\Delta t}{\sqrt{\overline{\varepsilon}\nu}}, \frac{K_4\Delta t}{h}, 1\right)h^{m+1}\right), \tag{4.5}$$

$$\|u - u_h\|_{l^\infty(0,t_n;\mathbf{H}^1(D))} \le C\left(h^{\omega_2} + \Delta t^2 + \frac{\sqrt{t_n}}{\Delta t}\min\left(\frac{K_4\Delta t}{\sqrt{\overline{\varepsilon}\nu}}, \frac{K_4\Delta t}{h}, 1\right)h^{m+1}\right), \tag{4.6}$$

where $K_4 = \|u\|_{L^\infty(\mathbf{L}^\infty(D))}$, $\omega_1 = \min(m+1, 2)$, $\omega_2 = \min(m, 2)$, *and the constant* C *is of the form* $C = \nu^{-1}K\exp(\kappa t_n)$ *with* $K(u, p, t_n)$ *and* κ *being constants and* κ *independent of* Δt, h, *and* n.

It is easy to see that the induction hypotheses are satisfied for $n = 0$. Now, assuming that (4.5) and (4.6) hold for $n = N-1$, we have to prove that they are also true for $n = N$; we postpone this proof to section 4.2. An important consequence of the induction hypothesis (4.6) is that (see [18]) there is a constant $h_1 \in (0, h_s)$ independent of Δt and n such that

$$\Delta t\, |u_h^n|_{\mathbf{W}^{1,\infty}(D)} \le \varepsilon_d(h) < 1 \qquad \forall h \in (0, h_1]. \tag{4.7}$$

This can be proved by setting $|u_h^n|_{\mathbf{W}^{1,\infty}(D)} = |u^n - u_h^n|_{\mathbf{W}^{1,\infty}(D)} + |u^n|_{\mathbf{W}^{1,\infty}(D)}$ and using the inverse inequality (P4) to get

$$|u_h^n|_{\mathbf{W}^{1,\infty}(D)} \le Ch^{-d/2}\|u^n - u_h^n\|_{\mathbf{H}^1(D)} + |u^n|_{\mathbf{W}^{1,\infty}(D)}.$$

Now, by the induction hypothesis (4.6) and taking $\omega_2 = m = 1$ (the worst case)

$$\Delta t|u_h^n|_{\mathbf{W}^{1,\infty}(D)} \le C(\Delta t h^{1-d/2} + \Delta t^3 h^{-d/2}) + \Delta t|u^n|_{\mathbf{W}^{1,\infty}(D)}.$$

With $\Delta t = o(h^{d/4})$ it follows that $\varepsilon_d(h) \le Ch^{d/4}(1 + h^{1-d/2}) < 1$ for h sufficiently small.

4.1. Approximation of the departure points. We use the bound (4.7) to estimate the error in the approximation of the points $X^{n-l,n+1}(x)$ by both $X_h^{n-l,n+1}(x)$ and $\widetilde{X}_h^{n-l,n+1}(x)$. The points $X_h^{n-l,n+1}(x)$ are the numerical solution of (3.4) calculated by a numerical method of order $r \ge 2$. Then for any interval $I_n := [t_{n-l}, t_{n+1}]$ we can set

$$X(x, t_{n+1}; t_{n-l}) = x - \Delta t\psi(u, x, t_{n-l}, \Delta t) + R_n \tag{4.8}$$

and

$$X_h(x, t_{n+1}; t_{n-l}) = x - \Delta t\psi(u_h, x, t_{n-l}, \Delta t), \tag{4.9}$$

where R_n is a remainder such that when $u(t)$ is sufficiently smooth,

$$\|R_n\|_{\mathbf{L}^2(D)} \le C_9 \|D_t^r u\|_{L^2(t_{n-l},t_{n+1};\mathbf{L}^2(D))} \Delta t^{\frac{2r+1}{2}};$$

here the constant C_9 is independent of Δt, h, and n. The function ψ is the so-called increment function in the theory of numerical methods for ODEs; see [16]. The function ψ satisfies the following relations:

(Ψ1) $\psi(0, x, t_n, \Delta t) = 0$.

(Ψ2) For $x, y \in \overline{D}$, $x \neq y$, and for all n,

$$|\psi(v, x, t_n, \Delta t) - \psi(v, y, t_n, \Delta t)| \le C_{10} \|v\|_{L^\infty(\mathbf{W}^{1,\infty}(D))} |x - y|,$$

where $|\cdot|$ denotes the Euclidean distance and C_{10} is a positive constant depending on the coefficients of the numerical method and Δt, but such that when $\Delta t \to 0$, $C_{10} \to K$, with K being another constant independent of Δt.

(Ψ3) There exists a positive constant C_{11} independent of Δt and h, but depending on the coefficients of the method, such that for all n,

$$\|\psi(u, x, t_n, \Delta t) - \psi(u_h, x, t_n, \Delta t)\|_{\mathbf{L}^2(D)} \le C_{11} \sum_{j=n-l-r+1}^{n} \left\| u^j - u_h^j \right\|_{\mathbf{L}^2(D)}.$$

Lemma 4.3. *Taking Δt and h sufficiently small, the following statements hold:* (1) *the mapping $x \to X_h^{n-l,n+1}(x)$ is, for all n, a quasi-isometric homeomorphism of D onto D;* (2) *for all n and j, $1 \le j \le NE$, the mappings $F_j^{n-l,n+1} : \widehat{T} \to T_j^{n-l,n+1}$ and $F_{hj}^{n-l,n+1} : \widehat{T} \to T_{hj}^{n-l,n+1}$ are of class $\mathbf{C}^{1,1}$ and $\mathbf{C}^{0,1}$, respectively;* (3) *for all n, the mappings $x \to \widetilde{X}^{n-l,n+1}(x)$ and $x \to \widetilde{X}_h^{n-l,n+1}(x)$ are quasi-isometric homeomorphisms of $\overline{D}$ onto $\overline{D}$.*

Proof. (1) We note that, by virtue of (4.9) and the properties of ψ, $X_h^{n-l,n+1}(x)$ is a Lipschitz continuous mapping; moreover, $X_h^{n-l,n+1}(x)$ is injective because for $x \neq y$ it follows from (4.9), (Ψ2), and (4.7) that

$$(1 - C_{10}\varepsilon_d(h)) |x - y| \le \left| X_h^{n-l,n+1}(x) - X_h^{n-l,n+1}(y) \right| \le (1 + C_{10}\varepsilon_d(h)) |x - y|;$$

hence, $X_h^{n-l,n+1}(x) \neq X_h^{n-l,n+1}(y)$. The surjectivity is proved as in [18].

(2) We show that $F_j^{n-l,n+1}$ is of class $\mathbf{C}^{1,1}$. Noting that $F_j^{n-l,n+1}(\widehat{x}) = X^{n-l,n+1} \circ F_j(\widehat{x})$ and using the integral form of the solution of (2.3a) we can write

$$F_j^{n-l,n+1}(\widehat{x}) = \mathbf{B}_j \widehat{x} + \mathbf{b}_j - \int_{t_{n-l}}^{t_{n+1}} u(X(F_j(\widehat{x}), t_{n+1}; t), t) dt.$$

Since $u \in L^\infty(\mathbf{W}^{2,\infty}(D))$, by applying Lemma 2.2 it follows that $F_j^{n-l,n+1}$ is of class $C^{1,1}$. To prove that $F_{hj}^{n-l,n+1}$ is of class $\mathbf{C}^{0,1}$ we take $\widehat{x}_1$, $\widehat{x}_2 \in \widehat{T}$, $\widehat{x}_1 \neq \widehat{x}_2$, and then, recalling that $F_{hj}^{n-l,n+1}(\widehat{x}) = X_h^{n-l,n+1} \circ F_j(\widehat{x})$, we have by virtue of (4.9), (3.1), (4.7), and (Ψ2) a bounded constant L independent of $\widehat{x_1}$ and $\widehat{x_2}$ such that

$$\left| F_{hj}^{n-l,n+1}(\widehat{x}_1) - F_{hj}^{n-l,n+1}(\widehat{x}_2) \right| \le L |\widehat{x}_1 - \widehat{x}_2|,$$

where $L = \left(1 + C_{10}\Delta t \sum_{j=n-l-r+1}^{n} \|u_h^j\|_{\mathbf{W}^{1,\infty}(D)}\right) \|\mathbf{B}_j\|_{l^2}$ is a local Lipschitz constant; here $\|\mathbf{B}_j\|_{l^2}$ denotes the vector induced Euclidean matrix norm. Thus, this proves that $F_{hj}^{n-l,n+1}(\widehat{x})$ is Lipschitz continuous.

(3) We write the proof for $\widetilde{X}^{n-l,n+1}(x)$ because the proof for $\widetilde{X}_h^{n-l,n+1}(x)$ is essentially the same. Recalling Remark 3.2, for each T_j there exists one and only one element $\widetilde{T}_j^{n-l,n+1} = \widetilde{X}^{n-l,n+1}(T_j)$ such that $\overline{D} = \cup_j \widetilde{T}_j^{n-l,n+1}$. Next we note that the restrictions $\widetilde{X}_j^{n-l,n+1}(x)$ of $\widetilde{X}^{n-l,n+1}(x)$ on T_j, which can be expressed as

$$\widetilde{X}_j^{n-l,n+1}(x) = \widetilde{F}_j^{n-l,n+1} \circ F_j^{-1}(x) = \widetilde{\mathbf{B}}_j^{n-l,n+1}\mathbf{B}_j^{-1}(x - \mathbf{b}_j) + \widetilde{\mathbf{b}}_j^{n-l,n+1},$$

are quasi-isometric homeomorphisms of T_j onto $\widetilde{T}_j^{n-l,n+1}$ because $\widetilde{\mathbf{B}}_j^{n-l,n+1}$ and $\mathbf{B}_j^{-1}$ are $d \times d$ nonsingular matrices, and given x_1, $x_2 \in T_j$, $x_1 \neq x_2$,

$$\begin{aligned} \left\| \mathbf{B}_j \left(\widetilde{\mathbf{B}}_j^{n-l,n+1} \right)^{-1} \right\|_{l^2} |x_1 - x_2| &\leq \left| \widetilde{X}_j^{n-l,n+1}(x_1) - \widetilde{X}_j^{n-l,n+1}(x_2) \right| \\ &\leq \ \left\| \widetilde{\mathbf{B}}_j^{n-l,n+1}\mathbf{B}_j^{-1} \right\|_{l^2} |x_1 - x_2| . \end{aligned}$$

Similarly, for $\widetilde{X}_{hj}^{n-l,n+1}$ we have

$$\begin{aligned} \left\| \mathbf{B}_j \left(\widetilde{\mathbf{B}}_{hj}^{n-l,n+1} \right)^{-1} \right\|_{l^2} |x_1 - x_2| &\leq \left| \widetilde{X}_{hj}^{n-l,n+1}(x_1) - \widetilde{X}_{hj}^{n-l,n+1}(x_2) \right| \\ &\leq \ \left\| \widetilde{\mathbf{B}}_{hj}^{n-l,n+1}\mathbf{B}_j^{-1} \right\|_{l^2} |x_1 - x_2| \end{aligned}$$

so that, noting that when $x \in T_j$, $\widetilde{X}^{n-l,n+1}(x) = \widetilde{X}_j^{n-l,n+1}(x)$, then when $y \in \widetilde{T}_j^{n-l,n+1}$ we can set $(\widetilde{X}^{n-l,n+1})^{-1}(y) = (\widetilde{X}_j^{n-l,n+1})^{-1}(y)$. Next, we prove that $(\widetilde{X}^{n-l,n+1})^{-1}(y)$ and $\widetilde{X}^{n-l,n+1}(x)$ are continuous mappings. Considering that D_h is a regular partition, then

$$T_j \cap T_k = \begin{cases} \emptyset \ \text{ or} \\ \Gamma_{jk} \ \text{ or} \\ P_{jk} \ \text{ a vertex} \end{cases} \Longrightarrow \widetilde{T}_j^{n-l,n+1} \cap \widetilde{T}_k^{n-l,n+1} = \begin{cases} \emptyset \ \text{ or} \\ \widetilde{\Gamma}_{jk}^{n-l,n+1} \ \text{ or} \\ \widetilde{P}_{jk}^{n-l,n+1} \ \text{ a vertex,} \end{cases}$$

where $\widetilde{\Gamma}_{jk}^{n-l,n+1} = \widetilde{X}^{n-l,n+1}(\Gamma_{jk})$, so that for all $\overline{x} \in \Gamma_{jk}$, $\widetilde{X}_j^{n-l,n+1}(\overline{x}) = \widetilde{X}_k^{n-l,n+1}(\overline{x})$; similarly, for all $\overline{y} \in \widetilde{\Gamma}_{jk}^{n-l,n+1}$, there is one and only one $\overline{x} \in \Gamma_{jk}$ such that $\overline{x} = (\widetilde{X}_j^{n-l,n+1})^{-1}(\overline{y}) = (\widetilde{X}_k^{n-l,n+1})^{-1}(\overline{y})$, and consequently the mappings $\widetilde{X}^{n-l,n+1}(x)$ and $(\widetilde{X}^{n-l,n+1})^{-1}(y)$ are continuous. To end the proof we bound $\|\widetilde{\mathbf{B}}_{hj}^{n-l,n+1}\mathbf{B}_j^{-1}\|_{l^2}$ and $\|\widetilde{\mathbf{B}}_j^{n-l,n+1}\mathbf{B}_j^{-1}\|_{l^2}$, noting that by virtue of (2.3b) and (4.9) we can set

$$\widetilde{\mathbf{B}}_j^{n-l,n+1} = \mathbf{B}_j - \widetilde{\mathbf{G}}_j^{n-l,n+1} \ \text{ and } \ \widetilde{\mathbf{B}}_{hj}^{n-l,n+1} = \mathbf{B}_j - \Delta t \widetilde{\mathbf{G}}_{hj}^{n-l,n+1},$$

where $\widetilde{\mathbf{G}}_j^{n-l,n+1}$ and $\widetilde{\mathbf{G}}_{hj}^{n-l,n+1} \in \mathcal{L}(\mathbb{R}^d)$; for instance, when $d = 2$, $\widetilde{\mathbf{G}}_j^{n-l,n+1} = (g_{pq})$ and $\widetilde{\mathbf{G}}_{hj}^{n-l,n+1} = (g_{hpq})$, $1 \leq p$, $q \leq 2$, where

$$g_{pq} = \int_{t_{n-l}}^{t_{n+1}} \left(u_p(X(a_{q+1}^{(j)}, t_{n+1}; t), t) - u_p(X(a_1^{(j)}, t_{n+1}; t), t) \right) dt$$

and

$$g_{hpq} = \psi_p(u_h, a_{q+1}^{(j)}, t_{n-l}, \Delta t) - \psi_p(u_h, a_1^{(j)}, t_{n-l}, \Delta t).$$

Thus

$$\widetilde{\mathbf{B}}_j^{n-l,n+1}\mathbf{B}_j^{-1} = \mathbf{I} - \widetilde{\mathbf{G}}_j^{n-l,n+1}\mathbf{B}_j^{-1} \quad \text{and} \quad \widetilde{\mathbf{B}}_{hj}^{n-l,n+1}\mathbf{B}_j^{-1} = \mathbf{I} - \Delta t\widetilde{\mathbf{G}}_{hj}^{n-l,n+1}\mathbf{B}_j^{-1}.$$

Hence

$$\left\|\widetilde{\mathbf{B}}_j^{n-l,n+1}\mathbf{B}_j^{-1}\right\|_{l^2} \leq 1 + \left\|\widetilde{\mathbf{G}}_j^{n-l,n+1}\right\|_{l^2}\left\|\mathbf{B}_j^{-1}\right\|_{l^2} \quad \text{and}$$

$$\left\|\widetilde{\mathbf{B}}_{hj}^{n-l,n+1}\mathbf{B}_j^{-1}\right\|_{l^2} \leq 1 + \Delta t\left\|\widetilde{\mathbf{G}}_{hj}^{n-l,n+1}\right\|_{l^2}\left\|\mathbf{B}_j^{-1}\right\|_{l^2}.$$

Next, to bound $\|\widetilde{\mathbf{G}}_j^{n-l,n+1}\|_{l^2}$ we use together the inequality $\|\mathbf{A}\|_{l^2}^2 \leq \|\mathbf{A}\|_{l^1}\|\mathbf{A}\|_{l^\infty}$, the equivalence of norms in finite dimensional spaces, and Lemma 2.1 so that we find a constant $C(d)$ such that

$$\left\|\widetilde{\mathbf{G}}_j^{n-l,n+1}\right\|_{l^2} \leq (l+1)C\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{1,\infty}(D))}\,h_j.$$

Since $\left\|\mathbf{B}_j^{-1}\right\|_{l^2} \leq \frac{\widehat{h}}{\rho_j}$, where ρ_j is the supremum of the diameters of the spheres inscribed in T_j, and the mesh is quasi-uniformly regular; i.e., for $1 \leq j \leq NE$, there is a constant $\sigma \geq \frac{h}{\rho_j}$, there exists another constant $c_1 = c_1(l,d,\sigma)$ such that

$$\left\|\widetilde{\mathbf{G}}_j^{n-l,n+1}\right\|_{l^2}\left\|\mathbf{B}_j^{-1}\right\|_{l^2} \leq c_1\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{1,\infty}(D))}.$$

Similarly, using the property $(\Psi 2)$ and (4.7) we find a constant $c_2 = c_2(l,d,r,\sigma)$ such that

$$\Delta t\left\|\widetilde{\mathbf{G}}_{hj}^{n-l,n+1}\right\|_{l^2}\left\|\mathbf{B}_j^{-1}\right\|_{l^2} \leq c_2\varepsilon_d(h), \quad h \in (0,h_1).$$

Taking h and Δt sufficiently small such that $c_2\varepsilon_d(h) < 1$ and $c_1\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{1,\infty}(D))} < 1$, and using the inequality $\|(\mathbf{I}-\mathbf{A})^{-1}\| \geq \frac{1}{1+\|\mathbf{A}\|}$ when $\|\mathbf{A}\| < 1$, it follows that

$$\left\|\mathbf{B}_j\left(\widetilde{\mathbf{B}}_j^{n-l,n+1}\right)^{-1}\right\|_{l^2} \geq \frac{1}{1 + c_1\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{1,\infty}(D))}} \quad \text{and}$$

$$\left\|\mathbf{B}_j\left(\widetilde{\mathbf{B}}_{hj}^{n-l,n+1}\right)^{-1}\right\|_{l^2} \geq \frac{1}{1 + c_2\varepsilon_d(h)}.$$

Employing all these bounds, we can write for any T_j and for any $x_1, x_2 \in T_j$

$$\begin{aligned}\left(1 - c_3\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{1,\infty}(D))}\right)|x_1 - x_2| &\leq \left|\widetilde{X}^{n-l,n+1}(x_1) - \widetilde{X}^{n-l,n+1}(x_2)\right| \\ &\leq \left(1 + c_1\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{1,\infty}(D))}\right)|x_1 - x_2|\end{aligned} \tag{4.10}$$

and

(4.11)

$$(1 - c_4\varepsilon_d(h))\,|x_1 - x_2| \leq \left|\widetilde{X}_h^{n-l,n+1}(x_1) - \widetilde{X}_h^{n-l,n+1}(x_2)\right| \leq (1 + c_2\varepsilon_d(h))\,|x_1 - x_2|,$$

where $1 - c_3\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{1,\infty}(D))} = (1 + c_1\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{1,\infty}(D))})^{-1}$ and $1 - c_4\varepsilon_d(h) = (1 + c_2\varepsilon_d(h))^{-1}$. Thus, the mappings $\widetilde{X}^{n-l,n+1}(x)$ and $\widetilde{X}_h^{n-l,n+1}(x)$ are quasi-isometric homeomorphisms. □

Our next concern is to estimate the error $X^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x)$.

LEMMA 4.4. *Let $r \geq 2$ be the order of the numerical scheme employed to integrate (3.4). Then there exist constants $C_{12}, \ldots, C_{16}$, independent of Δt and h, such that for any n the following estimates hold:*

$$\begin{aligned}(4.12)\qquad \| X^{n-l,n+1}(x) - X_h^{n-l,n+1}(x) \|_{\mathbf{L}^2(D)} &\leq C_{12}\Delta t \sum_{j=n-l-r+1}^{n} \left\|u^j - u_h^j\right\|_{\mathbf{L}^2(D)} \\ &+ C_{13}\,\|D_t^r u\|_{L^2(t_{n-l},t_{n+1};\mathbf{L}^2(D))}\,\Delta t^{\frac{2r+1}{2}},\ l = 0, 1,\end{aligned}$$

$$(4.13)\qquad \left\|X^{n-l,n+1}(x) - \widetilde{X}^{n-l,n+1}(x)\right\|_{\mathbf{L}^\infty(D)} \leq C_{14}h^2\Delta t\,\|u\|_{L^\infty(\mathbf{W}^{2,\infty}(D))},$$

$$(4.14)\qquad \left\|\widetilde{X}^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x)\right\|_{\mathbf{L}^2(D)} \leq C_{15}\left\|X^{n-l,n+1}(x) - X_h^{n-l,n+1}(x)\right\|_{\mathbf{L}^2(D)},$$

$$\begin{aligned}&\Delta t\sum_{n=1}^{k}\sum_{l=0}^{1}\left\|\frac{X^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x)}{\Delta t}\right\|_{\mathbf{L}^2(D)}^2 \\ (4.15)\qquad &\leq\ C_{16}\left\{h^{2\omega_1} + \Delta t^4 + t_k\left(\frac{h^{m+1}}{\Delta t}\right)^2 \min\left(\frac{K_4^2\Delta t^2}{\overline{\varepsilon}\nu}, \frac{K_4^2\Delta t^2}{h^2}, 1\right)\right. \\ &\qquad\left. +\ \Delta t^{2r}\,\|D_t^r u\|_{L^2(0,t_k;\mathbf{L}^2(D))}^2\right\}.\end{aligned}$$

Proof. (1) The estimate (4.12) follows from (4.8), (4.9), and condition (Ψ3) for the increment function ψ.

(2) Recalling the definitions of $X^{n-l,n+1}(x)$ and $\widetilde{X}^{n-l,n+1}(x)$, it is convenient to work with the mappings $F_j^{n-l,n+1}$ and $\widetilde{F}_j^{n-l,n+1}$ because for each T_j,

$$(4.16)\qquad \left\|X^{n-l,n+1}(x) - \widetilde{X}^{n-l,n+1}(x)\right\|_{\mathbf{L}^\infty(T_j)} = \left\|F_j^{n-l,n+1} - \widetilde{F}_j^{n-l,n+1}\right\|_{\mathbf{L}^\infty(\widehat{T})}.$$

From (3.5a), (3.5c), (3.5d), the integral form (2.3b) for $X^{n-1,n+1}(x)$, and approximation theory, it follows that

$$\begin{aligned}\left\|F_j^{n-l,n+1} - \widetilde{F}_j^{n-l,n+1}\right\|_{\mathbf{L}^\infty(\widehat{T})} &\leq \widehat{C}\left|F_j^{n-l,n+1}\right|_{\mathbf{W}^{2,\infty}(\widehat{T})} \\ &=\ \widehat{C}\left|\int_{t_{n-l}}^{t_{n+1}} u(X(F_j(\widehat{x}), t_{n+1}; t), t)dt\right|_{\mathbf{W}^{2,\infty}(\widehat{T})},\end{aligned}$$

where $|\cdot|_{\mathbf{W}^{2,\infty}(\widehat{T})^d}$ is the seminorm and $\widehat{C} = C(\widehat{I},\widehat{T})$. Next, noting that for $t \in [t_{n-l}, t_{n+1}]$, $X(\cdot, t_{n+1}; t) \circ F_j : \widehat{T} \to T_j^{t,n+1} := \{X(x, t_{n+1}; t) : x \in T_j\} \subset \overline{D}$, then making a change of variable and using Lemma 2.1 it is easy to see that there is a positive constant $C = C(T_j^{t,n+1})$ such that

$$\left| \int_{t_{n-l}}^{t_{n+1}} u(X(F_j(\widehat{x}), t_{n+1}; t), t)dt \right|_{\mathbf{W}^{2,\infty}(\widehat{T})} \leq C\Delta t h_j^2 \|u\|_{L^\infty(t_{n-l},t_{n+1};\mathbf{W}^{2,\infty}(T_j^{t,n+1}))}$$

$$\leq C_{14}\Delta t h^2 \|u\|_{L^\infty(\mathbf{W}^{2,\infty}(D))}.$$

From this inequality and (4.16) the result (4.13) follows.

(3) To prove (4.14) we set

$$\left\| \widetilde{X}^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x) \right\|_{\mathbf{L}^2(D)} = \sum_j \left\| \widetilde{X}^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x) \right\|_{\mathbf{L}^2(T_j)},$$

and by virtue of (3.5e), (3.5d), and (3.5a), denoting $|\mathbf{B}_j|$ the determinant of $\mathbf{B}_j$,

$$\left\| \widetilde{X}^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x) \right\|_{\mathbf{L}^2(T_j)} = |\mathbf{B}_j|^{1/2} \left\| \widehat{I}\left(F_j^{n-l,n+1} - F_{hj}^{n-l,n+1} \right) \right\|_{\mathbf{L}^2(\widehat{T})}$$

$$\leq C|\mathbf{B}_j|^{1/2} \left\| F_j^{n-l,n+1} - F_{hj}^{n-l,n+1} \right\|_{\mathbf{L}^2(\widehat{T})} = C \left\| X^{n-l,n+1}(x) - X_h^{n-l,n+1}(x) \right\|_{\mathbf{L}^2(T_j)}.$$

(4) To prove (4.15) we set $X^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x) = X^{n-l,n+1}(x) - \widetilde{X}^{n-l,n+1}(x) + \widetilde{X}^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x)$ and apply the above estimates and the induction hypothesis. □

We establish a lemma that we will need below, the proof of which can be achieved by using the equivalence of norms in finite dimensional spaces and the techniques of Lemma 4.3.

LEMMA 4.5. *Assume the induction hypotheses and* (4.7) *hold. Then there exists a constant $h_2 \in (0, h_1)$ independent of Δt and n such that for $h \in (0, h_2)$,*

$$x \to H_\alpha(x) = \alpha X^{n,n+1}(x) + (1-\alpha)\widetilde{X}_h^{n,n+1}(x), \quad 0 \leq \alpha \leq 1, \tag{4.17}$$

is a quasi-isometric homeomorphism of $\overline{D}$ onto $\overline{D}$.

LEMMA 4.6. *Let $v \in L^\infty(\mathbf{W}^{1,\infty}(D))$. At any time t_n, there exist constants C_{17} and C_{18} independent of Δt and $h \in (0, h_2)$ such that the following inequalities hold:*

$$\begin{aligned} &\left\| v^n(X^{n,n+1}(x)) - v^n(\widetilde{X}_h^{n,n+1}(x)) \right\|_{\mathbf{L}^2(D)} \\ &\leq C_{17} \left\| X^{n,n+1}(x) - \widetilde{X}_h^{n,n+1}(x) \right\|_{\mathbf{L}^2(D)} \|\nabla v^n\|_{\mathbf{L}^\infty(D)}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\left\| v^n(X^{n,n+1}(x)) - v^n(\widetilde{X}_h^{n,n+1}(x)) \right\|_{\mathbf{L}^1(D)} \\ &\leq C_{18} \left\| X^{n,n+1}(x) - \widetilde{X}_h^{n,n+1}(x) \right\|_{\mathbf{L}^2(D)} \|\nabla v^n\|_{\mathbf{L}^2(D)}. \end{aligned} \tag{4.19}$$

Proof. For $0 \leq \alpha \leq 1$, let $H_\alpha(x) = \alpha X^{n,n+1}(x) + (1-\alpha)\widetilde{X}_h^{n,n+1}(x)$ and

$$\mu(x) = \frac{H_1(x) - H_0(x)}{|H_1(x) - H_0(x)|},$$

so that we can write

$$\begin{aligned}
&\left| v^n(X^{n,n+1}(x)) - v^n(\widetilde{X}^{n,n+1}(x)) \right| \\
&\leq \left| X^{n,n+1}(x) - \widetilde{X}^{n,n+1}(x) \right| \int_0^1 \left| \frac{\partial v^n(H_\alpha(x))}{\partial \mu} \right| d\alpha.
\end{aligned}$$

Then

$$\begin{aligned}
&\left\| v^n(X^{n,n+1}(x)) - v^n(\widetilde{X}_h^{n,n+1}(x)) \right\|_{\mathbf{L}^1(D)} \\
&= \sum_j \int_{T_j} \left| X^{n,n+1}(x) - \widetilde{X}^{n,n+1}(x) \right| \int_0^1 \left| \frac{\partial v^n(H_\alpha(x))}{\partial \mu} \right| d\alpha dx \\
&\leq C \left(\sum_j \int_{T_j} \left| X^{n,n+1}(x) - \widetilde{X}^{n,n+1}(x) \right|^2 dx \right)^{1/2} \\
&\qquad \times \left(\int_0^1 \left(\sum_j \int_{H_\alpha(T_j)} |\nabla v^n(x)|^2 \, dx \right) d\alpha \right)^{1/2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left\| v^n(X^{n,n+1}(x)) - v^n(\widetilde{X}_h^{n,n+1}(x)) \right\|_{\mathbf{L}^2(D)} \\
&\leq C \left\| \nabla v^n \right\|_{\mathbf{L}^\infty(D)} \left(\sum_j \int_{T_j} \left| X^{n,n+1}(x) - \widetilde{X}^{n,n+1}(x) \right|^2 \right)^{1/2}.
\end{aligned}$$

Hence the estimates (4.18) and (4.19) are readily obtained. ∎

It is worth remarking that for all n,

$$\left\| u_h^n(X^{n,n+1}(x)) \right\|_{L^2(D)} = \| u_h^n \|_{L^2(D)} \tag{4.20}$$

because the Jacobian determinant of $x \to X^{n,n+1}(x)$ is equal to 1 a.e.

4.2. Error analysis.

4.2.1. Error analysis for the velocity in the L^2 and H^1 norms.

The error function for the velocity at time instant t_{n+1} is $u^{n+1}(x) - u_h^{n+1}(x)$, and we set

$$u^{n+1}(x) - u_h^{n+1}(x) = \rho^{n+1}(x) + \theta_h^{n+1}(x),$$

where $\rho^{n+1}(x)$ and $\theta_h^{n+1}(x)$ are defined as

$$\rho^{n+1}(x) := u^{n+1}(x) - w_h^{n+1}(x), \qquad \theta_h^{n+1}(x) := w_h^{n+1}(x) - u_h^{n+1}(x). \tag{4.21}$$

Making use of (4.1) it follows that for all $v_h \in \mathbf{V}_h$,

$$\nu(\nabla w_h^{n+1}, \nabla v_h) = -\left(\frac{Du}{Dt}\,|_{t=t_{n+1}}, v_h\right) + (f^{n+1}, v_h).$$

Taking $v_h \in \mathbf{V}_h$ in (3.3) and subtracting the latter equation yields

$$\begin{cases} \dfrac{3}{2}\left(u_h^{n+1}, v_h\right) - \Delta t\nu(\nabla\theta_h^{n+1}, \nabla v_h) = \Delta t\left(\dfrac{Du}{Dt}\,|_{t=t_{n+1}}, v_h\right) \\ \\ + 2\left(u_h^n(\widetilde{X}_h^{n,n+1}(x)), v_h\right) - \dfrac{1}{2}\left(u_h^{n-1}(\widetilde{X}_h^{n-1,n+1}(x)), v_h\right). \end{cases}$$

Introducing the notation

$$a^{*n-l} := a^{n-l}(X^{n-l,n-l+1}(x)), \quad a^{**n-l} := a^{n-l}(X^{n-l,n+1}(x)) \quad (l = 0, 1)$$

and using the relations

$$u_h^{n-l}(\widetilde{X}_h^{n-l,n+1}(x)) = u_h^{**n-l} - \left(u_h^{**n-l} - u_h^{n-l}(\widetilde{X}_h^{n-l,n+1}(x))\right) \text{ and}$$
$$u_h^{**n-l} = u^{**n-l} - \rho^{**n-l} - \theta_h^{**n-l},$$

the above equation yields

$$2\left(3\theta_h^{n+1} - 4\theta_h^{*n} + \theta_h^{**n-1}, v_h\right) + 4\Delta t\nu\left(\nabla\theta_h^{n+1}, \nabla v_h\right) = 4\sum_{i=1}^{7}\left(B_i, v_h\right), \tag{4.22}$$

where

$$B_1 = \Delta t\left(\frac{3u^{n+1} - 4u^{*n} + u^{**n-1}}{2\Delta t} - \frac{Du}{Dt}\,|_{t=t_{n+1}}\right), \quad B_2 = -\frac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{2},$$

$$B_3 = -2(\rho^n - \rho^{*n}), \quad B_4 = \frac{1}{2}\left(\rho^{n-1} - \rho^{**n-1}\right),$$

$$B_5 = -2\left(\rho^{*n} - u^{*n} - \left(\rho^n\left(\widetilde{X}_h^{n,n+1}(x)\right) - u^n\left(\widetilde{X}_h^{n,n+1}(x)\right)\right)\right),$$

$$B_6 = \frac{1}{2}\left(\rho^{**n-1} - u^{**n-1} - \left(\rho^{n-1}\left(\widetilde{X}_h^{n-1,n+1}(x)\right) - u^{n-1}\left(\widetilde{X}_h^{n-1,n+1}(x)\right)\right)\right),$$

$$B_7 = -2\left(\theta_h^{*n} - \theta_h^n\left(\widetilde{X}_h^{n,n+1}(x)\right)\right) + \frac{1}{2}\left(\theta_h^{**n-1} - \theta_h^{n-1}\left(\widetilde{X}_h^{n-1,n+1}(x)\right)\right).$$

To proceed with the analysis we need Lemma 7 of [3], which is an extension of Lemma 1 of [6].

LEMMA 4.7. *Let $l = 0, 1$ for all $n \geq l$, $\rho^{n-l}(x) - \rho^{n-l}(X^{n-l,n+1}(x))$ satisfy the following bounds:*

$$\left\|\rho^{n-l} - \rho^{n-l}(X^{n-l,n+1}(x))\right\|_{\mathbf{H}^{-1}} \leq (l+1)K_4\Delta t\left\|\rho^{n-l}\right\|_{\mathbf{L}^2(D)}, \tag{4.23a}$$

$$\left\|\rho^{n-l} - \rho^{n-l}(X^{n-l,n+1}(x))\right\|_{\mathbf{L}^2(D)} \leq (l+1)K_4\Delta t\left\|\nabla\rho^{n-l}\right\|_{\mathbf{L}^2(D)}, \tag{4.23b}$$

$$\left\|\rho^{n-l} - \rho^{n-l}(X^{n-l,n+1}(x))\right\|_{\mathbf{L}^2(D)} \leq 2\left\|\rho^{n-l}\right\|_{\mathbf{L}^2(D)}, \tag{4.23c}$$

where $K_4 = \|u\|_{L^\infty(\mathbf{L}^\infty(D))}$.

Proof. See Lemma 7 of [3]. ◻

As we will see below, each of the above bounds will yield a different term in the estimate of the error analysis. Returning to (4.22), we set $v_h = \theta_h^{n+1}$ and use the relation

$$2(3a^{n+1} - 4a^n + a^{n-1})a^{n+1} = \left|a^{n+1}\right|^2 + \left|2a^{n+1} - a^n\right|^2 - \left|a^n\right|^2 - \left|2a^n - a^{n-1}\right|^2 + \left|\delta_2 a^{n+1}\right|^2,$$

where $\{a^n\}$ is a sequence of real numbers and $\delta_2 a^{n+1} = a^{n+1} - 2a^n + a^{n-1}$, to get

$$\begin{cases} 2(3\theta_h^{n+1} - 4\theta_h^{*n} + \theta_h^{**n-1}, \theta_h^{n+1}) = \left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} + \left\|2\theta_h^{n+1} - \theta_h^{*n}\right\|^2_{\mathbf{L}^2(D)} \\ \qquad + \left\|\overline{\delta}_2\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} - \left\|\theta_h^{*n}\right\|^2_{\mathbf{L}^2(D)} - \left\|2\theta_h^{*n} - \theta_h^{**n-1}\right\|^2_{\mathbf{L}^2(D)}; \end{cases}$$

here $\overline{\delta}_2\theta_h^{n+1} = \theta_h^{n+1} - 2\theta_h^{*n} + \theta_h^{**n-1}$. Next, noting that $\|\theta_h^{*n}\|^2_{\mathbf{L}^2(D)} = \|\theta_h^n\|^2_{\mathbf{L}^2(D)}$ (see (4.20)), making the change of variable $y = X^{n,n+1}(x)$, and using the group property of the mapping $X(x, t_{n+1}; t)$ show that

$$\begin{aligned} \left\|2\theta_h^{*n} - \theta_h^{**n-1}\right\|^2_{\mathbf{L}^2(D)} &= \int_D \left|2\theta_h^n(X^{n,n+1}(x)) - \theta_h^{n-1}\left(X^{n-1,n+1}(x)\right)\right|^2 dx \\ &= \int_D \left|2\theta_h^n(y) - \theta_h^{n-1}\left(X^{n-1,n}(y)\right)\right|^2 dy = \left\|2\theta_h^n - \theta_h^{*n-1}\right\|^2_{\mathbf{L}^2(D)}, \end{aligned} \tag{4.24}$$

and then from (4.22) the inequality

$$\begin{cases} \left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} + \left\|2\theta_h^{n+1} - \theta_h^{*n}\right\|^2_{\mathbf{L}^2(D)} + 4\nu\Delta t\left\|\nabla\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} \\ \leq \left\|\theta_h^n\right\|^2_{\mathbf{L}^2(D)} + \left\|2\theta_h^n - \theta_h^{*n-1}\right\|^2_{\mathbf{L}^2(D)} + 4\sum_{i=1}^7 \left|(B_i, \theta_h^{n+1})\right| \end{cases} \tag{4.25}$$

follows. We are in position to state the convergence in the $l^\infty(\mathbf{L}^2(D))$ and $l^\infty(\mathbf{H}^1(D))$ norms.

THEOREM 4.8. *Let $\Delta t = o(h^{d/4})$ and assume that the following items hold: (1) the trajectories of the mesh points that are approximated by a numerical method of order $r \geq 2$; (2) the approximation properties* (P1)–(P4) *and the regularity hypotheses* (R1) *and* (R2)*; (3) the assumptions on the initial time steps* (4.4) *and the assumptions of Lemmas* 4.1 *and* 4.2*. Then, there exists a constant $\overline{\varepsilon} = O(T)$ such that for $h \in (0, h_2)$,*

$$\|u - u_h\|_{l^\infty(\mathbf{L}^2(D))} \leq C_{19}\left(h^{\omega_1} + \Delta t^2 + \min\left(\frac{K_4\Delta t}{\sqrt{\overline{\varepsilon}\nu}}, \frac{K_4\Delta t}{h}, 1\right)\frac{h^{m+1}}{\Delta t}\right) \tag{4.26}$$

and

$$\|u - u_h\|_{l^\infty(\mathbf{H}^1(D))} \leq C_{20}\left(h^{\omega_2} + \Delta t^2 + \min\left(\frac{K_4\Delta t}{\sqrt{\overline{\varepsilon}\nu}}, \frac{K_4\Delta t}{h}, 1\right)\frac{h^{m+1}}{\Delta t}\right), \tag{4.27}$$

where $K_4 = \|u\|_{L^\infty(\mathbf{L}^\infty(D))}$, $\omega_1 = \min(m+1, 2)$, $\omega_2 = \min(m, 2)$, and the constants C_{19} and C_{20} are of the form $C_{19} = K_1\exp(\kappa_2 T)$ and $C_{20} = K_2\exp(K^*T)$, with $K_1(u, p, T, \nu^{-1})$, $K_2(u, p, T, \nu^{-1})$, K^* and κ_2 being positive constants, κ_2 independent of Δt, h, u, p, and ν^{-1}, and K^* depending on $\|\nabla u\|_{L^\infty(\mathbf{L}^\infty(D))}$.

Remark 4.9. Before proving the theorem it is worth considering this remark. Following the proof of the theorem it is easy to see that the LG methods exhibit the same convergence behavior as the MLG methods, with the exception that in the LG methods $\omega_1 = m+1$ and $\omega_2 = m$, factor 2 is specific of the MLG methods and is a consequence of approximating the curved simplices $T^{n-l,n+1}$ by the straight ones $\widetilde{T}^{n-l,n+1}$. According to (4.26) we have the following scenario, which is supported by numerical experiments to be published elsewhere:

(A) Given h and Δt, the errors of both the LG– and MLG–BDF2 methods are of the form $O(\max(h^{\omega_1}, \Delta t^2))$ whether (A1) $(K_4\Delta t/h) < 1$ and $1/\sqrt{\nu} \leq h^{-1}$, or (A2) $(K_4\Delta t/h) > 1$ and $(K_4\Delta t/\sqrt{\bar{\varepsilon}\nu}) < 1$. This case may occur at moderate or low Reynolds numbers.

(B) Given h and Δt such that $(K_4\Delta t/h) < 1$ and $(K_4\Delta t/\sqrt{\bar{\varepsilon}\nu}) \geq (K_4\Delta t/h)$, the error is $O(\max(h^{\omega_1}, h^m, \Delta t^2))$. Notice that in the LG–BDF2 method the space error is h^m instead of h^{m+1}; this means that the space error is not optimal. This case occurs when Δt is sufficiently small.

(C) Given h and Δt such that $(K_4\Delta t/h) > 1$ and $(K_4\Delta t/\sqrt{\bar{\varepsilon}\nu}) > 1$, the error is $O(\max(h^{\omega_1}, h^{m+1}/\Delta t, \Delta t^2))$. This last case occurs when Δt is sufficiently large.

Proof. As we mention above (see induction hypotheses (IH)), we prove the theorem by induction on n. First, we note that by virtue of (4.4) the estimates (4.26) and (4.27) are true for $n = 0$. Now, assuming that (4.26) and (4.27) hold for $0 < t_n \leq t_{N-1}$, we shall prove that they also hold when $t_n = t_N = T$. To do so, we estimate the terms $\left|\left(B_i, \theta_h^{n+1}\right)\right|$ in (4.25).

By the Cauchy–Schwarz inequality,

$$\left|\left(B_1, \theta_h^{n+1}\right)\right| \leq \|B_1\|_{\mathbf{L}^2(D)} \left\|\theta_h^{n+1}\right\|_{\mathbf{L}^2(D)}.$$

To estimate $\|B_1\|_{\mathbf{L}^2(D)} \left\|\theta_h^{n+1}\right\|_{\mathbf{L}^2(D)}$ we first perform a Taylor series expansion along the curves $X(x, t_{n+1}; t)$ and then apply the elementary inequality $ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$, with ε being a positive real constant. Thus, with $c_1 = (2+2\sqrt{2})/\sqrt{5}$,

$$\left\| \frac{3u^{n+1} - 4u^{*n} + u^{**n-1}}{2\Delta t} - \frac{Du}{Dt}\; |_{t=t_{n+1}} \right\|_{\mathbf{L}^2(D)} \leq \Delta t^{3/2} c_1 \left\| \frac{D^3 u}{Dt^3} \right\|_{L^2(t_{n-1}, t_{n+1}; \mathbf{L}^2(D))}.$$

Hence,

$$4\left|\left(B_1, \theta_h^{n+1}\right)\right| \leq \frac{\Delta t^4}{\varepsilon} \frac{8(1+\sqrt{2})^2}{5} \left\|D_t^3 u\right\|^2_{L^2(t_{n-1}, t_{n+1}; \mathbf{L}^2(D))} + 2\varepsilon\Delta t \left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}. \tag{4.28}$$

As for the term $\left|\left(B_2, \theta_h^{n+1}\right)\right|$, again the Cauchy–Schwarz inequality yields

$$\left|\left(B_2, \theta_h^{n+1}\right)\right| \leq \|B_2\|_{\mathbf{L}^2(D)} \left\|\theta_h^{n+1}\right\|_{\mathbf{L}^2(D)}.$$

To bound $\|B_2\|_{\mathbf{L}^2(D)}$ we note that

$$\left| \frac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{2} \right| \leq \frac{3}{2} \int_{t_{n-1}}^{t_{n+1}} |\rho_t|\, dt \leq \frac{3}{2} \Delta t^{1/2} \left(\int_{t_{n-1}}^{t_{n+1}} |\rho_t|^2\, dt \right)^{1/2}.$$

Hence,

$$\|B_2\|_{\mathbf{L}^2(D)} \leq \frac{3}{2}\Delta t^{1/2} \|\rho_t\|_{L^2(t_{n-1},t_{n+1};\mathbf{L}^2(D))} .$$

Applying the elementary inequality it follows that

$$4\left|\left(B_2, \theta_h^{n+1}\right)\right| \leq \frac{9}{2\varepsilon} \|\rho_t\|^2_{L^2(t_{n-1},t_{n+1};\mathbf{L}^2(D))} + 2\varepsilon\Delta t \left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} . \tag{4.29}$$

To estimate $\left|\left(B_3, \theta_h^{n+1}\right)\right|$ and $\left|\left(B_4, \theta_h^{n+1}\right)\right|$ we apply Lemma 4.7 and in this way get three different estimates. We start with $\left|\left(B_3, \theta_h^{n+1}\right)\right|$ and notice that

$$\left|\left(B_3, \theta_h^{n+1}\right)\right| \leq \begin{cases} \|B_3\|_{\mathbf{H}^{-1}} \left\|\theta_h^{n+1}\right\|_{\mathbf{H}^1(D)} \quad \text{or} \\ \\ \|B_3\|_{\mathbf{L}^2(D)} \left\|\theta_h^{n+1}\right\|_{\mathbf{L}^2(D)} . \end{cases}$$

(A) By virtue of the first inequality of Lemma 4.7,

$$\|B_3\|_{\mathbf{H}^{-1}} = 2\left\|\rho^n - \rho^n(X^{n,n+1}(x))\right\|_{\mathbf{H}^{-1}} \leq 2K_4\Delta t \|\rho^n\|_{\mathbf{L}^2(D)} .$$

(B) By virtue of the second inequality of Lemma 4.7,

$$\|B_3\|_{\mathbf{L}^2(D)} \leq 2K_4\Delta t \|\nabla\rho^n\|_{\mathbf{L}^2(D)} .$$

(C) Finally, by virtue of the third inequality,

$$\|B_3\|_{\mathbf{L}^2(D)} \leq 2\|\rho^n\|_{\mathbf{L}^2(D)} .$$

Using these estimates and the elementary inequality, $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$, we have the following cases.

Case (A). Specifically for this case we set $\varepsilon = 2\nu\eta$, η being a positive number, calculated below, which is used to adjust the constants,

$$4\left|\left(B_3, \theta_h^{n+1}\right)\right| \leq \frac{4\Delta t K_4^2}{\nu\eta} \|\rho^n\|^2_{\mathbf{L}^2(D)} + 4\nu\eta\Delta t \left(\left\|\nabla\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} + \frac{1}{L_c^2}\left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}\right),$$

where L_c is a characteristic length which appears in the L^2 norm to make consistent the dimensional units.

Case (B). $4\left|\left(B_3, \theta_h^{n+1}\right)\right| \leq \frac{4\Delta t K_4^2}{\varepsilon} \|\nabla\rho^n\|^2_{\mathbf{L}^2(D)} + 4\varepsilon\Delta t \left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}$.

Case (C). $4\left|\left(B_3, \theta_h^{n+1}\right)\right| \leq \frac{4}{\varepsilon}\Delta t \|\frac{\rho^n}{\Delta t}\|^2_{\mathbf{L}^2(D)} + 4\varepsilon\Delta t \left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}$.

We proceed similarly with $4\left(B_4, \theta_h^{n+1}\right)$ Then we have

$$\begin{aligned} 4\left(\left|\left(B_3, \theta_h^{n+1}\right)\right| + \left|\left(B_4, \theta_h^{n+1}\right)\right|\right) &\leq \frac{6\Delta t K_4^2}{\nu\eta} \|\rho\|^2_{L^\infty(\mathbf{L}^2(D))} \\ &\quad + 6\nu\eta\Delta t \left(\left\|\nabla\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} + \frac{1}{L_c^2}\left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}\right), \end{aligned} \tag{4.30a}$$

(4.30b)
$$4\left(\left|\left(B_3,\theta_h^{n+1}\right)\right|+\left|\left(B_4,\theta_h^{n+1}\right)\right|\right)\leq\frac{6\Delta tK_4^2}{\varepsilon}\left\|\nabla\rho\right\|^2_{L^\infty(\mathbf{L}^2(D))}+6\varepsilon\Delta t\left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)},$$

(4.30c)
$$4\left(\left|\left(B_3,\theta_h^{n+1}\right)\right|+\left|\left(B_4,\theta_h^{n+1}\right)\right|\right)\leq\frac{6}{\varepsilon}\Delta t\left\|\frac{\rho}{\Delta t}\right\|^2_{L^\infty(\mathbf{L}^2(D))}+6\varepsilon\Delta t\left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}.$$

To bound $\left|\left(B_5,\theta_h^{n+1}\right)\right|$ we note that

$$\begin{aligned}\left|\left(B_5,\theta_h^{n+1}\right)\right|\leq 2\int_D\left|\rho^{*n}-\rho^n\left(\widetilde{X}_h^{n,n+1}(x)\right)\right|\left|\theta_h^{n+1}\right|dx\\ +\ 2\int_D\left|u^{*n}-u^n\left(\widetilde{X}_h^{n,n+1}(x)\right)\right|\left|\theta_h^{n+1}\right|dx.\end{aligned}$$

Using first the Cauchy–Schwarz inequality and then (4.18) and (4.3), there is a constant G independent of Δt and h such that

$$\begin{aligned}&2\int_D\left|\rho^{*n}-\rho^n\left(\widetilde{X}_h^{n,n+1}(x)\right)\right|\left|\theta_h^{n+1}\right|dx\\ \leq&\ G\left\|\theta_h^{n+1}\right\|_{\mathbf{L}^2(D)}\left\|X^{n,n+1}(x)-\widetilde{X}_h^{n,n+1}(x)\right\|_{\mathbf{L}^2(D)}.\end{aligned}$$

Similarly,

$$\left\{\begin{aligned}&2\int_D\left|u^{*n}-u^n\left(\widetilde{X}_h^{n,n+1}(x)\right)\right|\left|\theta_h^{n+1}\right|dx\\ &\leq\ 2\left\|u^{*n}-u^n\left(\widetilde{X}_h^{n,n+1}(x)\right)\right\|_{\mathbf{L}^2(D)}\left\|\theta_h^{n+1}\right\|_{\mathbf{L}^2(D)}\\ &\leq\ 2C_{19}\left\|\theta_h^{n+1}\right\|_{\mathbf{L}^2(D)}\left\|X^{n,n+1}(x)-\widetilde{X}_h^{n,n+1}(x)\right\|_{\mathbf{L}^2(D)}\left\|\nabla u^n\right\|_{\mathbf{L}^\infty(D)}.\end{aligned}\right.$$

In the same way, we bound the components of the term $\left|\left(B_6,\theta_h^{n+1}\right)\right|$. Thus, collecting these two bounds we find there is another constant that we also denote by G such that

$$\begin{aligned}&4(\left|\left(B_5,\theta_h^{n+1}\right)\right|+\left|\left(B_6,\theta_h^{n+1}\right)\right|)\leq 5\epsilon\Delta t\left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}\\ &+\ \frac{5G}{\varepsilon}\Delta t\left\{\sum_{l=0}^{1}\left\|\frac{X^{n-l,n+1}(x)-\widetilde{X}_h^{n-l,n+1}(x)}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}\right\}.\end{aligned}\tag{4.31}$$

To estimate the term $\left|\left(B_7,\theta_h^{n+1}\right)\right|$ we note that

$$\left|\left(B_7,\theta_h^{n+1}\right)\right|\leq 2\left\|\theta_h^{n+1}\right\|_{\mathbf{L}^\infty(D)}\sum_{l=0}^{1}\left\|\theta_h^{**n-l}-\theta_h^{n-l}\left(\widetilde{X}_h^{n-l,n+1}(x)\right)\right\|_{\mathbf{L}^1(D)}.$$

By virtue of (4.19) and the inverse inequality (3.2e), the right-hand side of this inequality is bounded by the term

$$2\left\|\theta_h^{n+1}\right\|_{\mathbf{H}^1(D)}\left(\sum_{l=0}^{1} D(h)\left\|\nabla\theta_h^{n-l}\right\|_{\mathbf{L}^2(D)}\left\|X^{n-l,n+1}(x)-\widetilde{X}_h^{n-l,n+1}(x)\right\|_{\mathbf{L}^2(D)}\right).$$

Now, noting that $\left\|\nabla\theta_h^{n-l}\right\|_{\mathbf{L}^2(D)} \leq \left\|u^{n-l}-u_h^{n-l}\right\|_{\mathbf{H}^1(D)}+\left\|\rho^{n-l}\right\|_{\mathbf{H}^1(D)}$, we apply Lemma 4.1 to bound $\left\|\rho^{n-l}\right\|_{\mathbf{H}^1(D)}$ and the induction hypothesis (4.6), and obtain that $D(h)\left\|\theta_h^{n-l}\right\|_{\mathbf{H}^1(D)} \leq CD(h)(h^m+\Delta t^2) \to 0$ as $h \to 0$ because for $\Delta t = o(h^{d/4})$, $D(h)\Delta t^2 = o(h^{(d-1)/2})$; hence, there exists another constant G independent of $\Delta t,\ h$, and n such that we can set

$$\begin{aligned}4\left|\left(B_7,\theta_h^{n+1}\right)\right| &\leq \frac{\nu\eta}{2}\Delta t\left\|\nabla\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}\\ &+\frac{\Delta t}{2\nu\eta}G\sum_{l=0}^{1}\left\|\frac{X^{n-l,n+1}(x)-\widetilde{X}_h^{n-l,n+1}(x)}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}.\end{aligned} \tag{4.32}$$

Next, to simplify the expressions that follow, we introduce the notation

$$\begin{cases} A_1^{n+1} := \left\|2\theta_h^{n+1}-\theta_h^{*n}\right\|^2_{\mathbf{L}^2(D)} - \left\|2\theta_h^{n}-\theta_h^{*n-1}\right\|^2_{\mathbf{L}^2(D)}, \\ A_2^{n+1} := \nu\Delta t\left\|\nabla\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}, \\ A_3^{n+1} := \dfrac{8(1+\sqrt{2})^2}{5}\left\|D_t^3 u\right\|^2_{L^2(t_{n-1},t_{n+1};\mathbf{L}^2(D))}, \\ A_4^{n+1} := \left\|\rho_t\right\|^2_{L^2(t_{n-1},t_{n+1};\mathbf{L}^2(D))}, \\ b_1 := 1-9\varepsilon_1\Delta t - \dfrac{8\nu\Delta t}{3L_c^2},\ b_2 := 1-15\varepsilon_2\Delta t,\ b_3 := 1-19\varepsilon_3\Delta t. \end{cases}$$

To proceed further, we fix $\varepsilon_1 = O(T^{-1})$ such that $0 < b_1 \ll 1$ and choose ε_2 and ε_3 to have $b_2 = b_3 = b_1$; then, setting $\eta = 6/13$ and $\varepsilon = \varepsilon_1$ when using the estimates (4.28), (4.29), (4.30a), (4.31), and (4.32), or $\eta = 6$ and $\varepsilon = \varepsilon_2$ when using both sequences of estimates {(4.28), (4.29), (4.30b), (4.31), and (4.32)} and {(4.28), (4.29), (4.30c), (4.31), and (4.32)}, we get from (4.25) that

$$\begin{aligned}\left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} + \kappa_1(A_1^{n+1}+A_2^{n+1}) &\leq (1+\kappa_2\Delta t)\left\|\theta_h^{n}\right\|^2_{\mathbf{L}^2(D)} + \overline{\gamma}_1\Delta t^4 A_3^{n+1}\\ &+\overline{\gamma}_2 A_4^{n+1} + F\Delta t\sum_{l=0}^{1}\left\|\frac{X^{n-l,n+1}(x)-\widetilde{X}_h^{n-l,n+1}(x)}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}\\ &+\begin{cases}\kappa_1\dfrac{9}{\nu}K_4^2\Delta t\left\|\rho\right\|^2_{L^\infty(\mathbf{L}^2(D))}, \text{ or}\\ \kappa_1\dfrac{6}{\varepsilon_2}K_4^2\Delta t\left\|\nabla\rho\right\|^2_{L^\infty(\mathbf{L}^2(D))}, \text{ or}\\ \kappa_1\dfrac{190}{15\varepsilon_2}\Delta t\left\|\dfrac{\rho}{\Delta t}\right\|^2_{L^\infty(\mathbf{L}^2(D))},\end{cases}\end{aligned}$$

where

$$F := \max\kappa_1 G\left(\frac{5}{\varepsilon_1}+\frac{13}{12\nu}, \frac{19\times 5}{15\varepsilon_2}+\frac{1}{12\nu}\right), \qquad \kappa_1 := b_2^{-1} := (1+\kappa_2\Delta t),$$

$\overline{\gamma}_1 := \max(\kappa_1\varepsilon_1^{-1}, 19\kappa_1\varepsilon_2^{-1})$, $\overline{\gamma}_2 := \max(9\kappa_1\varepsilon_1^{-1}, \frac{171}{15}\kappa_1\varepsilon_2^{-1})$, and $\overline{\varepsilon} := \max(\varepsilon_1^{-1}, \varepsilon_2^{-1})$. Invoking Lemma 4.1 to bound $\|\rho\|^2_{L^\infty(\mathbf{L}^2(D))}$, $\|\nabla\rho\|^2_{L^\infty(\mathbf{L}^2(D))}$, and $\|\frac{\rho}{\Delta t}\|^2_{L^\infty(\mathbf{L}^2(D))}$ we can write

$$\begin{aligned}
\left\|\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)} + \kappa_1\left(A_1^{n+1} + A_2^{n+1}\right) &\le (1+\kappa_2\Delta t)\left\|\theta_h^n\right\|^2_{\mathbf{L}^2(D)} + \overline{\gamma}_1\Delta t^4 A_3^{n+1} \\
&\quad + \overline{\gamma}_2 A_4^{n+1} + F\Delta t\sum_{l=0}^{1}\left\|\frac{X^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x)}{\Delta t}\right\|^2_{\mathbf{L}^2(D)} \\
&\quad + \frac{190}{15}\kappa_1\overline{\varepsilon}K\Delta t\left(\frac{h^{m+1}}{\Delta t}\right)^2\min\left(\frac{135}{190}\frac{K_4^2\Delta t^2}{\overline{\varepsilon}\nu}, \frac{90}{190}\frac{K_4^2\Delta t^2}{h^2}, 1\right).
\end{aligned} \tag{4.33}$$

Using (4.15), using Lemma 4.1 to estimate A_4^{n+1}, and adding both sides of (4.33) from $n=1$ up to $N-1$, we get positive constants K, G_2, G_3, and G_4 independent of Δt, h, and n such that

$$\begin{aligned}
&\left\|\theta_h^N\right\|^2_{\mathbf{L}^2(D)} + \kappa_1\nu\Delta t\sum_{n=1}^{N-1}\left\|\nabla\theta_h^n\right\|^2_{\mathbf{L}^2(D)} + \kappa_1\left\|2\theta_h^N - \theta_h^{*N-1}\right\|^2_{\mathbf{L}^2(D)} \\
&\le \kappa_2\Delta t\sum_{n=1}^{N-1}\left\|\theta_h^n\right\|^2_{\mathbf{L}^2(D)} + 5\left\|\theta_h^1\right\|^2_{\mathbf{L}^2(D)} + \overline{\gamma}_2 K h^{2(m+1)}\left\|u_t\right\|^2_{L^2(\mathbf{L}^2(D))} \\
&\quad + G_2 T\left\{\left(h^{m+1} + \Delta t^2\right)^2 + h^4\right\} + \overline{\gamma}_1\Delta t^4\left\|D_t^3\right\|^2_{L^2(\mathbf{L}^2(D))} \\
&\quad + G_3\Delta t^{2r}\left\|D_t^r u\right\|^2_{L^2(\mathbf{L}^2(D)} \\
&\quad + \frac{190}{15}\kappa_1\overline{\varepsilon}G_4 T\min\left(\frac{K_4^2\Delta t^2}{\overline{\varepsilon}\nu}, \frac{K_4^2\Delta t^2}{h^2}, 1\right)\left(\frac{h^{m+1}}{\Delta t}\right)^2.
\end{aligned} \tag{4.34}$$

Applying the Gronwall inequality, Lemma 4.1, and the initial estimate (4.4) to bound $\left\|\theta_h^1\right\|_{\mathbf{L}^2(D)}$, the result (4.26) follows.

To prove the result (4.27) we set $v_h = \frac{\theta_h^{n+1} - \theta_h^{*n}}{\Delta t}$ in (4.22) and use the relations $2ab = a^2 + b^2 - (a-b)^2$, $2a(a-b) = a^2 - b^2 + (a-b)^2$ and the inequality $ab \le (\varepsilon/2)a^2 + (1/2\varepsilon)b^2$, with $a, b > 0$, and $\varepsilon = 1$, so that we obtain

(4.35)

$$\begin{aligned}
&\frac{\left\|\theta_h^{n+1} - \theta_h^{*n}\right\|^2_{\mathbf{L}^2(D)}}{\Delta t} + 2\Delta t\left\|\frac{\theta_h^{n+1} - \theta_h^{*n}}{\Delta t}\right\|^2_{\mathbf{L}^2(D)} + 2\nu\left\{\left\|\nabla\left(\theta_h^{n+1} - \theta_h^{*n}\right)\right\|^2_{\mathbf{L}^2(D)}\right. \\
&\quad + \left.\left\|\nabla\theta_h^{n+1}\right\|^2_{\mathbf{L}^2(D)}\right\} \le \frac{\left\|\theta_h^{*n} - \theta_h^{**n-1}\right\|^2_{\mathbf{L}^2(D)}}{\Delta t} + 2\nu\left\|\nabla\theta_h^{*n}\right\|^2_{\mathbf{L}^2(D)} \\
&\quad + \frac{2}{\Delta t}\left\|\sum_{i=1}^{6}B_i\right\|^2_{\mathbf{L}^2(D)} + 4\left|\left(B_7, \frac{\theta_h^{n+1} - \theta_h^{*n}}{\Delta t}\right)\right|.
\end{aligned}$$

Here, by virtue of Lemma 2.1 it follows that $\left\|\nabla\theta_h^{*n}\right\|^2_{\mathbf{L}^2(D)} \le (1 + K^*\Delta t)\left\|\nabla\theta_h^n\right\|^2_{\mathbf{L}^2(D)}$, where K^* is a constant independent of Δt, h, and n, but depending on $\|\nabla u\|_{L^\infty(\mathbf{L}^\infty(D))}$;

$\left\|\theta_h^{*n}-\theta_h^{**n-1}\right\|_{\mathbf{L}^2(D)}=\left\|\theta_h^n-\theta_h^{*n-1}\right\|_{\mathbf{L}^2(D)}$ as we show in (4.24); finally, the last term on the right is worked out as the bound (4.32). Now, adding both sides of (4.35) from $n=1$ up to $n=N-1$ it follows that there is a constant K independent of Δt and h, but depending on T and ν^{-1}, such that

(4.36)

$$\left\|\nabla\theta_h^N\right\|^2_{\mathbf{L}^2(D)}\leq K^*\Delta t\sum_{n=1}^{N-1}\left\|\nabla\theta_h^n\right\|^2_{\mathbf{L}^2(D)}+\left\|\nabla\theta_h^1\right\|^2_{\mathbf{L}^2(D)}+\frac{1}{2\Delta t}\left\|\theta_h^1-\theta_h^{*0}\right\|^2_{\mathbf{L}^2(D)}$$
$$+\sum_{n=1}^{N-1}12\Delta t\sum_{i=1}^{6}\left\|\frac{B_i}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}+K\sum_{l=0}^{1}\left\|\frac{X^{n-l,n+1}(x)-\widetilde{X}_h^{n-l,n+1}(x)}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}.$$

The second and third terms on the right-hand side of this inequality are bounded by using the initial hypothesis (4.4) and Lemma 4.1, so that there is a constant K independent of Δt and h such that

$$\frac{1}{2\Delta t}\left\|\theta_h^1-\theta_h^{*0}\right\|_{\mathbf{L}^2(D)}+\left\|\nabla\theta_h^1\right\|_{\mathbf{L}^2(D)}\leq K\left(h^m+\Delta t^3\right). \tag{4.37}$$

As for the fourth term, we see from previous calculations that there is another constant K such that

$$\begin{aligned}\sum_{n=1}^{N-1}\Delta t\sum_{i=1}^{6}\left\|\frac{B_i}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}\leq K\Bigg\{&\Delta t^4\left\|D_t^3\right\|^2_{L^2(\mathbf{L}^2(D))}+\left\|\rho_t\right\|^2_{L^2(\mathbf{L}^2(D))}\\&+\left\|\nabla\rho\right\|^2_{L^\infty(\mathbf{L}^2(D))}+\sum_{l=0}^{1}\left\|\frac{X^{n-l,n+1}(x)-\widetilde{X}_h^{n-l,n+1}(x)}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}\Bigg\}.\end{aligned} \tag{4.38}$$

Next, we substitute (4.38) and (4.37) into (4.36) and use the Gronwall inequality, (4.15), and Lemma 4.1 to get the estimate (4.27). ∎

4.2.2. Error analysis for the pressure in the L^2 norm. The error function for the pressure at time t_n is defined as $p^n-p_h^n:=\omega^n+\chi_h^n$, where $\omega^n:=p^n-\mu_h^n$ and $\chi_h:=\mu_h^n-p_h^n$, so that

$$\left\|p-p_h\right\|_{l^2(L^2(D))}\leq\left\|\omega\right\|_{L^2(L^2(D))}+\left\|\chi_h\right\|_{l^2(L^2(D))}.$$

THEOREM 4.10. *Under the assumptions of Theorem* 4.8 *there exists a positive constant* $C_{21}(u,p,\nu^{-1},\beta^{-1},T)$ *such that*

$$\left\|p-p_h\right\|_{l^2(L^2(D))}\leq C_{21}\left(h^{\omega_2}+\Delta t^2+\min\left(\frac{K_4\Delta t}{\sqrt{\varepsilon}\nu},\frac{K_4\Delta t}{h},1\right)\frac{h^{m+1}}{\Delta t}\right). \tag{4.39}$$

Proof. $\left\|\omega\right\|_{L^2(L^2(D))}$ is estimated by Lemma 4.1, so that it remains to estimate χ_h. To do so, we first notice that letting $v_h\in\mathbf{X}_h$ in (4.1) and (4.22) yields

$$\nu\left(\nabla w_h^{n+1},\nabla v_h\right)=-\left(D_tu\mid_{t=t_{n+1}},v_h\right)+\left(\mu_h^{n+1},\operatorname{div}\,v_h\right)+\left(f^{n+1},v_h\right)$$

and

$$\frac{1}{2}\left(3\theta_h^{n+1}-4\theta_h^{*n}+\theta_h^{**n-1},v_h\right)+\Delta t\nu\left(\nabla\theta_h^{n+1},\nabla v_h\right)=\sum_{i=1}^{7}\left(B_i,v_h\right)+\Delta t\left(\chi_h^{n+1},\operatorname{div}\,v_h\right).$$

Setting $v_h = \theta_h^{n+1}$ and applying (3.2a), the Poincaré inequality, and the arguments to bound $|(B_7, \theta_h^{n+1})|$, we find a constant $K(\beta)$ such that

$$K\Delta t \left\|\chi_h^{n+1}\right\|_{L^2(D)} \leq 3\left\|\theta_h^{n+1} - \theta_h^{*n}\right\|_{\mathbf{L}^2(D)} + \left\|\theta_h^{*n} - \theta_h^{**n-1}\right\|_{\mathbf{L}^2(D)}$$

$$+ \Delta t\nu \left\|\nabla\theta_h^{n+1}\right\|_{\mathbf{L}^2(D)} + 2\sum_{i=1}^{6}\|B_i\|_{\mathbf{L}^2(D)}$$

$$+ G\sum_{l=0}^{1}\left\|X^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x)\right\|_{\mathbf{L}^2(D)}.$$

Then, there exists a positive constant C independent of Δt, h, and n such that

(4.40)

$$\|p - p_h\|^2_{l^2(L^2(D))} \leq 2\|\omega\|^2_{L^2(L^2(D))} + 2\Delta t\sum_{n=1}^{N-1}\left\|\chi_h^{n+1}\right\|^2_{L^2(D)} \leq 2\|\omega\|_{L^2(L^2(D))}$$

$$+ C\left\{\sum_{n=1}^{N-1}\Delta t\sum_{i=1}^{6}\left\|\frac{B_i}{\Delta t}\right\|^2_{\mathbf{L}^2(D)} + \Delta t\sum_{l=0}^{1}\left\|\frac{X^{n-l,n+1}(x) - \widetilde{X}_h^{n-l,n+1}(x)}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}\right.$$

$$\left. + \|\theta_h\|^2_{l^\infty(\mathbf{H}^1(D))} + \Delta t\sum_{n=1}^{N-1}\left(\left\|\frac{\theta_h^{n+1} - \theta_h^{*n}}{\Delta t}\right\|^2_{\mathbf{L}^2(D)} + \left\|\frac{\theta_h^{*n} - \theta_h^{**n-1}}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}\right)\right\}.$$

From (4.35) it follows that

$$\Delta t\sum_{n=1}^{N-1}\left\|\frac{\theta_h^{n+1} - \theta_h^{*n}}{\Delta t}\right\|^2_{\mathbf{L}^2(D)} \leq \frac{1}{2\Delta t}\left\|\theta_h^1 - \theta_h^{*0}\right\|^2_{\mathbf{L}^2(D)}$$

$$+ \nu\left\|\theta_h^1\right\|^2_{\mathbf{H}^1(D)} + C\|\theta_h\|^2_{l^\infty(\mathbf{H}^1(D))} + \sum_{n=1}^{N-1}\Delta t\sum_{i=1}^{7}\left\|\frac{B_i}{\Delta t}\right\|^2_{\mathbf{L}^2(D)}.$$

A similar inequality holds for the term $\Delta t\sum_{n=1}^{N-1}\|\frac{\theta_h^{*n}-\theta_h^{**n-1}}{\Delta t}\|^2_{\mathbf{L}^2(D)}$ because as we know (see (4.24)), such a term is equal to $\Delta t\sum_{n=1}^{N-1}\|\frac{\theta_h^{n}-\theta_h^{*n-1}}{\Delta t}\|^2_{\mathbf{L}^2(D)}$. The estimate (4.39) is obtained by substituting theses inequalities into (4.40), applying (4.38), (4.15), (4.36) to bound $\left\|\theta_h^{n+1}\right\|^2_{l^2(\mathbf{H}^1(D))}$, and then applying Lemma 4.1 to bound the terms $\|\rho\|^2_{L^\infty(\mathbf{L}^2(D))}$ and $\|\rho_t\|^2_{L^2(\mathbf{L}^2(D))}$. $\square$

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